

# On the reflection of magnon bound states

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ABSTRACT: We investigate the reflection of two-particle bound states of a free open string in the light-cone  $AdS_5 \times S^5$  string sigma model, for large angular momentum  $J = J_{56}$  and ending on a D7 brane which wraps the entire  $AdS_5$  and a maximal  $S^3 \subset S^5$ . We use the superspace formalism to analyse fundamental and two-particle bound states in the cases of supersymmetry-preserving and broken-supersymmetry boundaries. We find the boundary  $S$ -matrices corresponding to bound states both in the bulk and on the boundary.

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## 1. Introduction

It has been recognized in recent years that the planar limit of  $\mathcal{N} = 4$  super Yang-Mills is integrable, and the  $S$ -matrix approach allows us to successfully study the spectra of superstrings propagating freely in  $AdS_5 \times S^5$  spacetime in the framework of the  $AdS/CFT$

correspondence conjectured by Maldacena et al. [1]. The  $S$ -matrix approach [2, 3, 4] was first developed in the spin chain framework in the perturbative regime of the gauge theory, where it allows one to conjecture the corresponding (all-loop) Bethe equations describing the asymptotic spectrum of the gauge theory [5, 6, 7]. Integrability allows one to find exact expressions for the  $S$ -matrices by requiring them to respect the underlying symmetries of the model. It is well-known that the  $S$ -matrix for the fundamental excitations in the bulk can be determined up to an overall (‘dressing’) phase factor from just the centrally extended  $\mathfrak{su}(2|2)$  symmetry [5, 8, 9], and the  $S$ -matrix so obtained respects the Yang-Baxter equation (YBE) and a generalized physical unitarity condition. The overall phase factor is severely constrained by crossing symmetry [10]. This non-analytic overall phase factor constitutes an important feature of the string  $S$ -matrix and has been the subject of intensive research [11, 12, 13, 14].

In the limit of infinite light-cone momentum, in addition to the fundamental particles, the spectrum of the string sigma model contains an infinite tower of bound states [15, 16, 17]. These manifest themselves as poles of the multi-particle  $S$ -matrix built from the fundamental  $S$ -matrix  $S^{AA}$ . More explicitly,  $l$ -particle bound states appear as the tensor product of two  $4l$ -dimensional atypical (short) totally symmetric multiplets of the centrally extended  $\mathfrak{su}(2|2)$  algebra [15, 18, 19]. This can be obtained from the  $l$ -fold tensor product of the fundamental multiplets by projecting it onto the totally symmetric component.

Construction of  $S$ -matrices for the bound states is more complicated, as the  $\mathfrak{su}(2|2)$  symmetry alone is no longer enough to determine the  $S$ -matrices uniquely. Further constraints are required, either from the YBE or the underlying Yangian symmetry [19]; one can then determine general  $l$ -particle bound state bulk  $S$ -matrices [20, 21, 22]. It is worth recalling that Yangians generically have some very nice properties, particularly at the level of representation theory [23, 24]. So the appearance of Yangian symmetry in the string context – for example, via the universal  $R$ -matrix [25, 26, 27] – is a very welcome feature.

As was shown in [20], the construction of the bound state  $S$ -matrix relies on the observation that the  $l$ -particle bound state representation  $\mathcal{V}_l$  of the centrally extended  $\mathfrak{su}(2|2)$  algebra may be realized on the space of homogeneous (super)symmetric polynomials of degree  $l$  depending on two bosonic and two fermionic variables,  $\omega_a$  and  $\theta_\alpha$  respectively. Thus, the representation space is identical to an irreducible short superfield  $\Phi^l(\omega, \theta)$ . In this realization the algebra generators are represented by differential operators  $\mathbb{J}$  linear in variables  $\omega_a$  and  $\theta_\alpha$  with the scattering coefficients being functions of the parameters describing the representation. The introduction of a space  $\mathcal{D}_l$  dual to  $\mathcal{V}_l$ , which may be realized as the space of differential operators preserving the homogeneous gradation of  $\Phi^l(\omega, \theta)$ , allows one to define the  $S$ -matrix as an element of

$$\text{End}(\mathcal{V}^{l_1} \otimes \mathcal{V}^{l_2}) \approx \mathcal{V}^{l_1} \otimes \mathcal{V}^{l_2} \otimes \mathcal{D}_{l_1} \otimes \mathcal{D}_{l_2}.$$

Thus the  $S$ -matrix  $S^{l_1 l_2}$  may be written as a differential operator of degree  $l_1 + l_2$  acting on the product of two superfields  $\Phi^{l_1}(\omega, \theta)$  and  $\Phi^{l_2}(\omega, \theta)$ .

The  $S$ -matrices  $S^{AB}$  and  $S^{BB}$  which describe the scattering processes involving the fundamental multiplet  $A$  and the two-particle bound state multiplet  $B$  were found in [20]. The invariance conditions for the latter only partially determine the scattering coefficients  $a_i$ , for if the tensor product  $\mathcal{V}^{l_1} \otimes \mathcal{V}^{l_2}$  has  $m$  irreducible components, then  $m - 1$  coefficients  $a_i$  together with an overall scale are left undetermined. The YBE turns out to be sufficient to determine the so-far-unrestricted  $m - 1$  coefficients  $a_i$ , leaving only the overall scale.

However, the underlying Yangian symmetry provides an alternative way of finding these coefficients [22], an approach which goes back to the inception of quantum groups [28]. This leads to a general strategy for finding the  $S$ -matrices of higher order bound states [22], which is essential since the fusion procedure does not work straightforwardly for AdS/CFT  $S$ -matrices [20]. These higher-order  $S$ -matrices play an important role in understanding the underlying integrability and deriving the transfer matrices, Bethe ansatz equations, and so on.

Very similar considerations apply to the analysis of the spectra of strings with open boundary conditions in the limit of infinite light-cone momentum [29, 30, 31, 32]. The reflection of fundamental magnons from a boundary was considered in [29, 32, 33], while the reflection of magnon bound states was considered in [34], and the Yangian symmetry of an open string attached to the giant graviton brane was recently exploited in [35]. The first and key question is to determine whether a particular boundary condition is integrable or not. It was shown in [36, 37] that working to first-order in the 't Hooft coupling the D7 brane yields integrable boundary conditions at least in the  $\mathfrak{so}(6)$  sector. Further investigations followed [38, 39, 40].

Deep in the bulk of an open spin chain the theories are indistinguishable from pure  $\mathcal{N} = 4$ , so the symmetry arguments discussed above remain valid, and the bulk  $S$ -matrix may be used without modifications. The task is then to determine the reflection of magnons off the end of the chain, where the residual symmetries of the boundary are crucial in determining the structure of the reflection  $K$ -matrix. It was shown in [29] that the relative orientation between the preferred  $R$ -charge of the vacuum and the spherical factor of the brane worldvolume affect the symmetries preserved by the reflection, and there are two inequivalent possibilities for reflection from D5 and D7 branes; further, only in certain cases can the boundary itself have an excitation attached to it. The nested coordinate Bethe ansatz equations for D3 (maximal giant graviton) and D7 branes were recently proposed in [41].

In this paper we consider the so-called ‘ $Z = 0$  D7-brane’ system, in which the usual gauge/string correspondence in  $AdS_5 \times S^5$  has a D7-brane wrapping the entire  $AdS_5$  and a maximal  $S^3$  of the  $S^5$  (defined by setting  $X^5 = X^6 = 0$ ) with a  $\mathcal{N} = 2$  super Yang-Mills theory living on it [29]. The preferred  $R$ -charge is  $J = J_{56}$  and we are considering states in which both  $J$  and the classical scaling dimension  $\Delta$  are large, but keeping the difference  $\Delta - J$  finite. Hence the vacuum state  $Z$  has  $\Delta - J = 0$  and the elementary magnons are the excitations with  $\Delta - J = 1$ .

Our main goal is to fill a gap in the literature on the reflection of bound states, particularly reflection from the D7 brane. The  $K$ -matrices  $K^{Aa}$  and  $K^{A1}$  which describe the scattering of the fundamental bulk multiplet  $A$  off the fundamental boundary multiplet  $a$  and singlet state 1 were found in [29], but the bound state reflection matrices are unknown. Our goal is to find the  $K$ -matrices  $K^{Ba}$ ,  $K^{Ab}$ ,  $K^{Bb}$  and  $K^{B1}$  using the superspace formalism presented in [20], where we denote the two particle bound state multiplet on the boundary as  $b$ . We choose the phase  $\zeta$  increasing from left to right in accordance with [19, 29, 33], but in contrast to [20]. Thus we shall need to calculate the bulk bound state  $S$ -matrices independently in order to check that our  $K$ -matrices satisfy the YBE.

The outline of this paper is as follows. In section 2 we briefly recall the details of the scattering of fundamental magnons, in the bulk and on the boundary. In section 3 we review the representation algebra of magnon bound states, recall the superspace formalism introduced in [20], and extend it to the boundary algebra. In sections 4 and 5 we present the description of  $S$ - and  $K$ - matrices in the superspace formalism. The results of our calculations, which involve some large sets of coefficients, are presented in appendices.

## 2. Symmetries and fundamental representations

In this section we shall briefly review the symmetries in the bulk and on the boundary. We shall consider the superconformal algebra  $\mathfrak{psu}(4|4)$  of the  $\mathcal{N} = 4$  SYM in the bulk [2] and  $\mathcal{N} = 2$  SYM on the boundary [42]. We build the scattering theory whose vacuum state is the operator  $\text{tr} Z^L$  with  $L \gg 1$  for the closed boundary conditions and the operator

$$\epsilon_{j_1, \dots, j_N}^{i_1, \dots, i_N} Z_{i_1}^{j_1} \dots Z_{j_{N-1}}^{j_{L-1}} (\chi_L Z^J \chi_R)_{i_N}^{j_N}, \quad (2.1)$$

for the open boundary conditions, where  $\chi_L$ ,  $\chi_R$  are the excitations living on the left and on the right boundaries and  $Z = \Phi_5 + i\Phi_6$  is the vacuum reference state with the charge under  $\Delta - J$  being zero. All remaining states have  $\Delta - J > 0$ .

### 2.1 Bulk case

The bulk superconformal algebra is  $\mathfrak{psu}(4|4) \cong \mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2)$ . We shall use the undotted and dotted indices to distinguish left and right Lorentz generators  $\mathbb{L}_\alpha^\beta \in \mathfrak{psu}(2|2)$ ,  $\tilde{\mathbb{L}}_{\dot{\alpha}}^{\dot{\beta}} \in \widetilde{\mathfrak{psu}}(2|2)$ , where

$$\alpha, \beta, \dots = +, -, \quad \dot{\alpha}, \dot{\beta}, \dots = \dot{+}, \dot{-}, \quad (2.2)$$

and  $\mathbb{R}_a^b$ , where

$$a, b, \dots = 1, 2, 3, 4, \quad (2.3)$$

to denote R-symmetry generators. The same notation will be used for supersymmetry generators  $\mathbb{Q}_\beta^b$ ,  $\mathbb{G}_b^\beta$  and  $\tilde{\mathbb{Q}}_{\dot{\beta}}^{\dot{b}}$ ,  $\tilde{\mathbb{G}}_{\dot{b}}^{\dot{\beta}}$ . The supercharges transform canonically according the indices they carry:

$$\begin{aligned}
[\mathbb{L}_\alpha^\beta, \mathbb{J}^\gamma] &= \delta_\alpha^\gamma \mathbb{J}^\beta - \frac{1}{2} \delta_\alpha^\beta \mathbb{J}^\gamma, & [\mathbb{L}_\alpha^\beta, \mathbb{J}_\gamma] &= -\delta_\gamma^\beta \mathbb{J}_\alpha + \frac{1}{2} \delta_\alpha^\beta \mathbb{J}_\gamma, \\
[\tilde{\mathbb{L}}_{\dot{\alpha}}^{\dot{\beta}}, \mathbb{J}^{\dot{\gamma}}] &= \delta_{\dot{\alpha}}^{\dot{\gamma}} \mathbb{J}^{\dot{\beta}} - \frac{1}{2} \delta_{\dot{\alpha}}^{\dot{\beta}} \mathbb{J}^{\dot{\gamma}}, & [\tilde{\mathbb{L}}_{\dot{\alpha}}^{\dot{\beta}}, \mathbb{J}_{\dot{\gamma}}] &= -\delta_{\dot{\gamma}}^{\dot{\beta}} \mathbb{J}_{\dot{\alpha}} + \frac{1}{2} \delta_{\dot{\alpha}}^{\dot{\beta}} \mathbb{J}_{\dot{\gamma}}, \\
[\mathbb{R}_a^b, \mathbb{J}^c] &= \delta_a^c \mathbb{J}^b - \frac{1}{4} \delta_a^b \mathbb{J}^c, & [\mathbb{R}_a^b, \mathbb{J}_c] &= -\delta_c^b \mathbb{J}_a + \frac{1}{4} \delta_a^b \mathbb{J}_c.
\end{aligned} \tag{2.4}$$

We shall be considering the subsectors  $\mathfrak{psu}(2|2)$  and  $\widetilde{\mathfrak{psu}}(2|2)$  of the whole symmetry separately. The relevant algebra shall be centrally extended  $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$  which we shall denote as  $\mathfrak{psu}(2|2)_C$  [5]. It is generated by the bosonic rotation generators  $\mathbb{L}_\alpha^\beta$ ,  $\mathbb{R}_a^b$ , the supersymmetry generators  $\mathbb{Q}_\beta^a$ ,  $\mathbb{G}_b^\beta$ , and three central charges  $\mathbb{H}$ ,  $\mathbb{C}$  and  $\mathbb{C}^\dagger$  obeying the relations

$$\begin{aligned}
\{\mathbb{Q}_\alpha^a, \mathbb{Q}_\beta^b\} &= \epsilon^{ab} \epsilon_{\alpha\beta} \mathbb{C}, \\
\{\mathbb{G}_a^\alpha, \mathbb{G}_b^\beta\} &= \epsilon^{\alpha\beta} \epsilon_{ab} \mathbb{C}^\dagger, \\
\{\mathbb{Q}_\alpha^a, \mathbb{G}_b^\beta\} &= \delta_b^a \mathbb{L}_\beta^\alpha + \delta_\beta^a \mathbb{R}_b^a + \delta_b^a \delta_\beta^\alpha \mathbb{H},
\end{aligned} \tag{2.5}$$

where  $a, b, \dots = 1, 2$  and  $\alpha, \beta, \dots = 3, 4$ . We shall be using this notation throughout remaining of the paper.

The fundamental excitations propagating in the bulk transform in the fundamental representation  $\square$  of the  $\mathfrak{psu}(2|2)_C$  and the bulk scattering matrix factors as a tensor product  $S \otimes \tilde{S}$ , where each factor acts as

$$S/\tilde{S} : \square \otimes \square \rightarrow \square \otimes \square. \tag{2.6}$$

The basis of the space consists a two of bosons  $\phi_a$  transforming as a doublet under  $su(2)_\mathbb{R}$  and two fermions  $\psi_\alpha$  - a doublet under  $su(2)_\mathbb{L}$ . The  $\mathfrak{psu}(2|2)_C$  supercharges act on this basis in the following way:

$$\begin{aligned}
\mathbb{Q}_\beta^b |\phi_a\rangle &= a \delta_a^b |\psi_\beta\rangle, & \mathbb{G}_b^\beta |\phi_a\rangle &= c \epsilon^{\beta\alpha} \epsilon_{ba} |\psi_\alpha\rangle, \\
\mathbb{Q}_\beta^b |\psi_\alpha\rangle &= b \epsilon^{ba} \epsilon_{\beta\alpha} |\phi_a\rangle, & \mathbb{G}_b^\beta |\psi_\alpha\rangle &= d \delta_\alpha^\beta |\phi_b\rangle.
\end{aligned} \tag{2.7}$$

The coefficients  $a, b, c, d$  respect the multiplet shortening condition  $ad - bc = 1$  and are parametrized as [8]

$$a = \sqrt{\frac{g}{2}} \eta, \quad b = \sqrt{\frac{g}{2}} \frac{i\zeta}{\eta} \left( \frac{x^+}{x^-} - 1 \right), \quad c = -\sqrt{\frac{g}{2}} \frac{\eta}{\zeta x^+}, \quad d = -\sqrt{\frac{g}{2}} \frac{x^+}{i\eta} \left( \frac{x^-}{x^+} - 1 \right), \tag{2.8}$$

where  $\zeta^1$  is an overall phase factor,  $\eta$  reflects the freedom of the choice of spectral parameters  $x^\pm$  obeying

$$e^{ip} = \frac{x^+}{x^-}, \quad x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}. \tag{2.9}$$

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<sup>1</sup>Our definition of the parameter  $\zeta$  should be replaced by  $\zeta \mapsto i\zeta$  for consistency with [5].

We shall talk about the particular choice of  $\zeta$  and  $\eta$  in the next section.

The values of the central charges for the fundamental multiplet are

$$\begin{aligned} C &= ab = \frac{i}{2}g (e^{ip} - 1) e^{2i\xi}, \\ C^\dagger &= cd = -\frac{i}{2}g (e^{-ip} - 1) e^{-2i\xi}, \\ H &= ad + bc = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}, \end{aligned} \quad (2.10)$$

where  $H$  gives the energy-momentum dispersion relation of the states.

## 2.2 Boundary case

We shall consider the so-called ‘ $Z = 0$  D7-brane’ system, where the brane is wrapping the entire  $AdS_5$  and a maximal  $S^3 \subset S^5$ . This case was nicely presented in [29]. Here we shall briefly review the properties of the configuration that are relevant to us.

The D7 brane is usually defined by setting  $X_5 = X_6 = 0$ . This choice breaks the  $\mathfrak{so}(6)$  R-symmetry down to  $\mathfrak{so}(4)_{1234} \times \mathfrak{so}(2)_{56}$ . It was shown in [42] that the presence of the D7-brane breaks the half of the background supersymmetries that are left handed with respect to the surviving  $\mathfrak{so}(4) \subset \mathfrak{so}(6)$  and the surviving fields on the brane form the  $\mathcal{N} = 2$  hypermultiplet. The choice of Bethe vacuum on the spin chain may further reduce the symmetries on the boundary. We shall consider the standard  $Z = X_5 + iX_6$  bulk vacuum case. The preferred R-charge  $J \equiv J_{56}$  rotates the directions transverse to the brane and preserves the full  $\mathfrak{so}(4)_{1234}$  R-symmetry, but breaks half of the supercharges, leaving the residual symmetry algebra on the boundary to be  $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \widetilde{\mathfrak{psu}}(2|2) \ltimes \mathbb{R}^3$ . This means that fundamental matter fields transform in a  $(1, \square)$  representation of  $\mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2)$ . It implies that the reflection matrix factors as a tensor product

$$K \otimes \tilde{K}, \quad (2.11)$$

where we have to consider two different reflection processes – the reflection from a supersymmetric boundary

$$\tilde{K} : \square \otimes \square \rightarrow \square \otimes \square, \quad (2.12)$$

and reflection from a singlet state on the boundary

$$K : \square \otimes 1 \rightarrow \square \otimes 1. \quad (2.13)$$

The fundamental representation of the excitations on the boundary is parametrized by the coefficients [33]

$$a_B = \sqrt{\frac{g}{2}}\eta_B, \quad b_B = -\sqrt{\frac{g}{2}}\frac{i\zeta}{\eta_B}, \quad c_B = -\sqrt{\frac{g}{2}}\frac{\eta_B}{\zeta x_B}, \quad d_B = \sqrt{\frac{g}{2}}\frac{x_B}{i\eta_B}, \quad (2.14)$$

and the shortening (mass-shell) condition reads as

$$x_B + \frac{1}{x_B} = \frac{2i}{g}. \quad (2.15)$$

The solution of the mass-shell condition

$$x_B = \frac{i}{g} \left( 1 + \sqrt{1 + g^2} \right), \quad (2.16)$$

is chosen to give a positive energy for the unexcited boundary state

$$\epsilon = ad + bc = \sqrt{1 + g^2}. \quad (2.17)$$

Note that the central charges  $C$  and  $C^\dagger$  are not conserved under the reflection, otherwise momentum would be preserved (only  $p \mapsto p$  would be allowed) leaving no sensible notion of reflection. Rather the total values of all three central charges are preserved under reflection and the outgoing momentum is indeed  $-p$  [29].

### 3. Magnon bound states

In this section we shall briefly discuss the representation structure of magnon bound states and the superspace formalism introduced in [20]. In this framework the  $S$ - and  $K$ - matrices are naturally realized as  $\mathfrak{su}(2) \otimes \mathfrak{su}(2)$ -invariant differential operators in the tensor product of two representations.

#### 3.1 The representation of bound states

$l$ -magnon bound states in the light-cone string theory on  $AdS_5 \times S^5$  are described by atypical totally symmetric representations of  $\mathfrak{su}(2|2)_C$ . The dimension of the representation is  $2l|2l$  and it can be realized on a graded vector space with the following basis:

- a tensor  $|e_{a_1 \dots a_l}\rangle$ , symmetric in  $a_i$  where  $a_i = 1, 2$  are bosonic indices, giving  $l + 1$  bosonic states;
- a tensor  $|e_{a_1 \dots a_{l-2} \alpha_1 \alpha_2}\rangle$ , symmetric in  $a_i$  and skew-symmetric in  $\alpha_i$  where  $\alpha_i = 3, 4$  are fermionic indices, giving  $l - 1$  bosonic states;
- a tensor  $|e_{a_1 \dots a_{l-1} \alpha}\rangle$ , symmetric in  $a_i$ , giving  $2l$  fermionic states.

The corresponding vector space is denoted as  $\mathcal{V}^l(p, \zeta)$ , where  $p$  and  $\zeta$  in general are complex parameters of the representation.



The action of the bosonic generators of the symmetry algebra on the basis of the corresponding vector space is

$$\begin{aligned}
\mathbb{L}_c^b |e_{a_1 \dots a_l}\rangle &= \delta_{a_1}^b |e_{c \dots a_l}\rangle + \dots + \delta_{a_M}^b |e_{a_1 \dots c}\rangle - \frac{l}{2} \delta_c^b |e_{a_1 \dots a_l}\rangle, \\
\mathbb{L}_c^b |e_{a_1 \dots a_{l-2} \alpha_1 \alpha_2}\rangle &= \delta_{a_1}^b |e_{c \dots a_{l-2} \alpha_1 \alpha_2}\rangle + \dots + \delta_{a_M}^b |e_{a_1 \dots c \alpha_1 \alpha_2}\rangle - \frac{l-2}{2} \delta_c^b |e_{a_1 \dots a_{l-2} \alpha_1 \alpha_2}\rangle, \\
\mathbb{L}_c^b |e_{a_1 \dots a_{l-1} \alpha}\rangle &= \delta_{a_1}^b |e_{c \dots a_{l-1} \alpha}\rangle + \dots + \delta_{a_M}^b |e_{a_1 \dots c \alpha}\rangle - \frac{l-1}{2} \delta_c^b |e_{a_1 \dots a_{l-1} \alpha}\rangle; \tag{3.1}
\end{aligned}$$

$$\begin{aligned}
\mathbb{R}_\gamma^\beta |e_{a_1 \dots a_l}\rangle &= 0, \\
\mathbb{R}_\gamma^\beta |e_{a_1 \dots a_{l-2} \alpha_1 \alpha_2}\rangle &= \delta_{a_1}^\beta |e_{a_1 \dots a_{l-2} \gamma \alpha_2}\rangle + \delta_{a_2}^\beta |e_{a_1 \dots c \alpha_1 \alpha_2}\rangle - \delta_\gamma^\beta |e_{a_1 \dots a_{l-2} \alpha_1 \alpha_2}\rangle, \\
\mathbb{R}_\gamma^\beta |e_{a_1 \dots a_{l-1} \alpha}\rangle &= \delta_\alpha^\beta |e_{a_1 \dots a_{l-1} \gamma}\rangle - \frac{1}{2} \delta_\gamma^\beta |e_{a_1 \dots a_{l-1} \alpha}\rangle; \tag{3.2}
\end{aligned}$$

while the action of the supersymmetric generators has the form

$$\begin{aligned}
\mathbb{Q}_\beta^b |e_{a_1 \dots a_l}\rangle &= a_1^l \left( \delta_{a_1}^b |e_{a_2 \dots a_l \alpha}\rangle + \dots + \delta_{a_l}^b |e_{a_1 \dots a_{l-1} \alpha}\rangle \right), \\
\mathbb{Q}_\beta^b |e_{a_1 \dots a_{l-2} \alpha_1 \alpha_2}\rangle &= b_2^l \epsilon^{b a_{l-1}} \left( \epsilon_{\beta \alpha_1} |e_{c \dots a_{l-1} \alpha_2}\rangle - \epsilon_{\beta \alpha_2} |e_{c \dots a_{l-1} \alpha_1}\rangle \right), \\
\mathbb{Q}_\beta^b |e_{a_1 \dots a_{l-1} \alpha}\rangle &= b_1^l \epsilon^{b a_l} \epsilon_{\beta \alpha} |e_{c \dots a_l}\rangle + a_2^l \left( \delta_{a_1}^b |e_{a_2 \dots a_{l-1} \beta \alpha}\rangle + \dots + \delta_{a_{l-1}}^b |e_{a_1 \dots a_{l-2} \beta \alpha}\rangle \right); \tag{3.3} \\
\mathbb{G}_b^\beta |e_{a_1 \dots a_l}\rangle &= c_1^l \epsilon^{\beta \alpha} \left( \epsilon_{b a_1} |e_{a_2 \dots a_l \alpha}\rangle + \dots + \epsilon_{b a_l} |e_{a_1 \dots a_{l-1} \alpha}\rangle \right), \\
\mathbb{G}_b^\beta |e_{a_1 \dots a_{l-2} \alpha_1 \alpha_2}\rangle &= d_2^l \left( \delta_{\alpha_1}^\beta |e_{a_1 \dots a_{l-2} b \alpha_2}\rangle - \delta_{\alpha_2}^\beta |e_{a_1 \dots a_{l-2} b \alpha_1}\rangle \right), \\
\mathbb{G}_b^\beta |e_{a_1 \dots a_{l-1} \alpha}\rangle &= d_1^l \delta_\alpha^\beta |e_{a_1 \dots a_{l-1} b}\rangle + c_2^l \epsilon^{\beta \gamma} \left( \epsilon_{b a_1} |e_{a_2 \dots a_{l-1} \gamma \alpha}\rangle + \dots + \epsilon_{b a_{l-1}} |e_{a_1 \dots a_{l-2} \gamma \alpha}\rangle \right). \tag{3.4}
\end{aligned}$$

The parameters  $a_i^l$ ,  $b_i^l$ ,  $c_i^l$ ,  $d_i^l$  are representation-dependent and may be determined by requiring that they respect the centrally extended  $\mathfrak{su}(2|2)_c$  algebra [5], which imposes the constraints

$$\begin{aligned}
b_1^l d_2^l &= b_2^l d_1^l, & c_1^l d_2^l &= c_2^l d_1^l, \\
a_1^l d_1^l - b_1^l c_1^l &= 1, & a_2^l d_2^l - b_2^l c_2^l &= 1. \tag{3.5}
\end{aligned}$$

The central charges obey the shortening condition

$$\mathbb{H}_l^2 - 4\mathbb{C}_l \mathbb{C}_l^\dagger = 1. \tag{3.6}$$

The eigenvalue of  $\mathbb{H}_l$  depends explicitly on the bound state number  $l$  in the following way

$$H_l = \sqrt{l^2 + 4g^2 \sin^2 \frac{p}{2}} = l \sqrt{1 + 4 \left( \frac{g}{l} \right)^2 \sin^2 \frac{p}{2}}, \tag{3.7}$$

where  $\frac{g}{l}$  may be called the effective coupling constant for an  $l$ -magnon bound state. In this way the values of  $\mathbb{C}_l$  and  $\mathbb{C}_l^\dagger$  may be defined to depend explicitly on  $l$  by setting

$$C_l = l \frac{i}{2} \frac{g}{l} (e^{ip} - 1) e^{2i\xi}, \quad C_l^\dagger = -l \frac{i}{2} \frac{g}{l} (e^{-ip} - 1) e^{-2i\xi}. \tag{3.8}$$

This yields the usual definition of central charges in terms of representation parameters,

$$\begin{aligned}\frac{C_l}{l} &= a_1^l d_1^l = a_2^l d_2^l, & \frac{C_l^\dagger}{l} &= c_1^l d_1^l = c_2^l d_2^l, \\ \frac{H_l}{l} &= \left( a_1^l d_1^l + b_1^l c_1^l \right) = \left( a_2^l d_2^l + b_2^l c_2^l \right),\end{aligned}\tag{3.9}$$

implying that it is always possible to choose  $a_i^l, b_i^l, c_i^l, d_i^l$  so that

$$a_1^l = a_2^l \equiv a, \quad b_1^l = b_2^l \equiv b, \quad c_1^l = c_2^l \equiv c, \quad d_1^l = d_2^l \equiv d\tag{3.10}$$

and thereby obtaining the the convenient parametrization

$$a = \sqrt{\frac{g}{2l}} \eta, \quad b = \sqrt{\frac{g}{2l}} \frac{i\zeta}{\eta} \left( \frac{x^+}{x^-} - 1 \right), \quad c = -\sqrt{\frac{g}{2l}} \frac{\eta}{\zeta x^+}, \quad d = -\sqrt{\frac{g}{2l}} \frac{x^+}{i\eta} \left( \frac{x^-}{x^+} - 1 \right),\tag{3.11}$$

where  $\zeta = e^{2i\xi}$  and the spectral parameters  $x^\pm$  respect the mass-shell condition of the  $l$ -magnon bound state,

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = i\frac{2l}{g}.\tag{3.12}$$

The conservation of central charges on the sum of an  $l$ - and an  $m$ -magnon bound state requires

$$\begin{aligned}C_{l+m} &= C_l + C_m \\ &= \frac{i}{2}g (e^{ip_1} - 1) e^{2i\xi_1} + \frac{i}{2}g (e^{ip_2} - 1) e^{2i\xi_2} \\ &= \frac{i}{2}g (e^{ip} - 1) e^{2i\xi_0},\end{aligned}\tag{3.13}$$

which is satisfied by setting the total momentum  $p = p_1 + p_2$  and  $\xi_1 \equiv \xi_0, \xi_2 \equiv \xi_0 + \frac{p_1}{2}$ . The same holds for  $C_l^\dagger$ .

The unitarity condition implies  $d^* = a, c^* = b$ , hence

$$\begin{aligned}\eta &= \left[ \frac{1}{i\eta} (x^+ - x^-) \right]^* = -\frac{i}{\eta^*} e^{i\varphi} (x^+ - x^-), \\ \frac{i\zeta}{\eta} \left( \frac{x^+}{x^-} - 1 \right) &= -\left[ \frac{\eta}{\zeta x^+} \right]^* = -\frac{\eta^*}{\zeta^* e^{i\varphi} x^-}.\end{aligned}\tag{3.14}$$

Here we have used the relation  $(x^\pm)^* = e^{i\varphi} x^\mp$ , where the phase factor  $e^{i\varphi}$  represents the freedom to choose the basis for  $x^\pm$ . These relations give

$$\begin{aligned}|\eta|^2 &= i e^{i\varphi} (x^- - x^+) \zeta, \\ \eta &= e^{i\xi} e^{i\frac{\varphi}{2}} \sqrt{i(x^- - x^+)}.\end{aligned}\tag{3.15}$$

Unitarity also implies that parameters  $\xi$  and  $\varphi$  must be real. Constraints on  $\xi$  were derived above, while  $\varphi$  may be chosen to acquire any value. The value  $\varphi = 0$  is commonly used for

the fundamental representation [33, 8, 29], while the value  $\varphi = \frac{p}{2}$  is preferred for the case of bound states [16].

The same considerations may be trivially extended for the representation of the boundary multi-magnon bound states. Thus the boundary representation of  $l$ -magnon bound states is described by the parameters

$$a_B = \sqrt{\frac{g}{2l}} \eta_B, \quad b_B = -\sqrt{\frac{g}{2l}} \frac{i\zeta}{\eta_B}, \quad c_B = -\sqrt{\frac{g}{2l}} \frac{\eta_B}{\zeta x_B}, \quad d_B = \sqrt{\frac{g}{2l}} \frac{x_B}{i\eta_B}, \quad (3.16)$$

and the shortening condition reads as

$$x_B + \frac{1}{x_B} = i \frac{2l}{g}. \quad (3.17)$$

The unitarity condition for the boundary representation gives

$$|\eta_B|^2 = -ix_B, \quad (3.18)$$

implying that boundary spectral parameter  $x_B$  is purely imaginary.

### 3.2 Superspace representation of $\mathfrak{su}(2|2)_C$

For a convenient description of the  $S$ -matrix a  $2l|2l$  graded vector space of monomials of degree  $l$  of two bosonic  $\omega_a$ ,  $a = 1, 2$ , and two fermionic variables  $\theta_\alpha$ ,  $\alpha = 3, 4$  may be introduced [20]. Then any homogeneous symmetric polynomial of degree  $l$  can be expressed as

$$\Phi_l(\omega, \theta) = \phi^{a_1 \dots a_l} \omega_{a_1} \dots \omega_{a_l} + \phi^{a_1 \dots a_{l-1} \alpha} \omega_{a_1} \dots \omega_{a_{l-1}} \theta_\alpha + \phi^{a_1 \dots a_{l-2} \alpha_1 \alpha_2} \omega_{a_1} \dots \omega_{a_{l-2}} \theta_{\alpha_1} \theta_{\alpha_2}.$$

The basis of monomials is related to the vector space of magnon bound states by

$$|m, n, \mu, \nu\rangle = N_{mn\mu\nu} \omega_1^m \omega_2^n \theta_3^\mu \theta_4^\nu, \quad (3.19)$$

where  $m, n \geq 0$ ,  $\mu, \nu = 0, 1$ ,  $m + n + \mu + \nu = l$  and the normalization constant is [20]

$$N_{mn\mu\nu} = \left( \frac{(l-1)!}{m! n!} \right)^{1/2}. \quad (3.20)$$

This basis is assumed to be orthogonal

$$\langle a, b, \alpha, \beta | m, n, \mu, \nu \rangle = \delta_{am} \delta_{bn} \delta_{\alpha\mu} \delta_{\beta\nu}. \quad (3.21)$$

The hermitian conjugate operators are expressed as

$$(\omega_a)^\dagger = \frac{\partial}{\partial \omega_a}, \quad (\theta_\alpha)^\dagger = \frac{\partial}{\partial \theta_\alpha} \quad (3.22)$$

and are considered to be real.

In this representation the centrally extended  $\mathfrak{su}(2|2)$  generators are realized as the differential operators

$$\begin{aligned}
\mathbb{L}_a{}^b &= \omega_a \frac{\partial}{\partial \omega_b} - \frac{1}{2} \delta_a^b \omega_c \frac{\partial}{\partial \omega_c}, \\
\mathbb{R}_\alpha{}^\beta &= \theta_\alpha \frac{\partial}{\partial \theta_\beta} - \frac{1}{2} \delta_\alpha^\beta \theta_\gamma \frac{\partial}{\partial \theta_\gamma}, \\
\mathbb{Q}_\alpha{}^a &= a \theta_\alpha \frac{\partial}{\partial \omega_a} + b \epsilon^{ab} \epsilon_{\alpha\beta} \omega_b \frac{\partial}{\partial \theta_\beta}, \\
\mathbb{G}_a{}^\alpha &= c \epsilon_{ab} \epsilon^{\alpha\beta} \theta_\beta \frac{\partial}{\partial \omega_b} + d \omega_a \frac{\partial}{\partial \theta_\alpha},
\end{aligned} \tag{3.23}$$

while the central charges are

$$\begin{aligned}
\mathbb{C} &= ab \left( \omega_a \frac{\partial}{\partial \omega_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), \\
\mathbb{C}^\dagger &= cd \left( \omega_a \frac{\partial}{\partial \omega_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right), \\
\mathbb{H} &= (ad + bc) \left( \omega_a \frac{\partial}{\partial \omega_a} + \theta_\alpha \frac{\partial}{\partial \theta_\alpha} \right).
\end{aligned} \tag{3.24}$$

### 3.3 $S$ - and $K$ - matrices in superspace formalism

The  $S$ -matrix in superspace is realized as a differential operator acting on the tensor product of two vector spaces

$$\mathcal{V}^M(p_1, \zeta_1) \otimes \mathcal{V}^N(p_2, \zeta_2) \sim \mathcal{V}^M(p_1, 1) \otimes \mathcal{V}^N(p_2, e^{ip_1}) \sim \mathcal{V}^M(p_1, e^{ip_2}) \otimes \mathcal{V}^N(p_2, 1).$$

The unitarity condition implies that there are two possible equivalent choices of phase factors,  $\zeta_1 = \zeta$ ,  $\zeta_2 = \zeta e^{ip_1}$  and  $\zeta_1 = \zeta e^{ip_2}$ ,  $\zeta_2 = \zeta$ . We define the  $S$ -matrix as an intertwining operator

$$S(p_1, p_2) : \mathcal{V}^M(p_1, \zeta) \otimes \mathcal{V}^N(p_2, \zeta e^{ip_1}) \rightarrow \mathcal{V}^M(p_1, \zeta e^{ip_2}) \otimes \mathcal{V}^N(p_2, \zeta),$$

where the phase  $\zeta$  increases from left to right. Our definition is consistent with that of Beisert et al.<sup>2</sup> but differs from that of Arutyunov et al., who choose the phase  $\zeta$  to increase from right to left

$$S(p_1, p_2) : \mathcal{V}^M(p_1, \zeta e^{ip_2}) \otimes \mathcal{V}^N(p_2, \zeta) \rightarrow \mathcal{V}^M(p_1, \zeta) \otimes \mathcal{V}^N(p_2, \zeta e^{ip_1}).$$

In this superspace formalism the  $S$ -matrix may be viewed as an element of

$$\text{End}(\mathcal{V}^M \otimes \mathcal{V}^N) \approx \mathcal{V}^M \otimes \mathcal{V}^N \otimes \mathcal{D}_M \otimes \mathcal{D}_N,$$

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<sup>2</sup>The usual  $S$ -matrix in the physical space  $S^{phys}$  is related to the  $S$ -matrix in the superspace  $S^{super}$  as  $S^{phys} = \mathcal{P} \cdot S^{super}$ , where  $\mathcal{P}$  is an ordinary graded permutation operator.

where  $\mathcal{D}_M$  is the vector space dual to  $\mathcal{V}^M$ . The dual vector space is realized as the space of polynomials of degree  $M$  of the differential operators  $\frac{\partial}{\partial \omega_a}$  and  $\frac{\partial}{\partial \theta_\alpha}$  with a natural pairing between  $\mathcal{D}_M$  and  $\mathcal{V}^M$  induced by the relations  $\frac{\partial}{\partial \omega_a} \omega_b = \delta_b^a$  and  $\frac{\partial}{\partial \theta_\alpha} \theta_\beta = \delta_\beta^\alpha$ . Thus the  $S$ -matrix may be represented as

$$S(p_1, p_2) = \sum_i a_i(p_1, p_2) \Lambda_i, \quad (3.25)$$

where  $\Lambda_i$  span a complete basis of differential operators invariant under the  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$  algebra and  $a_i(p_1, p_2)$  are  $S$ -matrix components. The exact expression for  $\Lambda_i$  for various  $S$ -matrices are given in [20].

Following the above analysis, we define the  $K$ -matrix describing the reflection of the bulk magnons from the boundary states as an operator acting on the tensor space in the following way:

$$K(p, q) : \mathcal{V}^M(p, \zeta) \otimes \mathcal{V}^N(q, \zeta e^{ip}) \rightarrow \mathcal{V}^M(-p, \zeta) \otimes \mathcal{V}^N(q, \zeta e^{2ip}),$$

where  $p$  is the momentum of the bulk state and  $q$  is some parameter describing the boundary state. Hence, the reflection matrix can be represented as

$$K(p, q) = \sum_i k_i(p, q) \Lambda_i, \quad (3.26)$$

where  $\Lambda_i$  have the same form as for bulk  $S$ -matrices.

## 4. $S$ -matrices

### $S$ -matrix $S^{AA}$

We define the  $S$ -matrix  $S^{AA}$  as an intertwining operator

$$S^{AA} : \mathcal{V}^1(p_1, \zeta) \otimes \mathcal{V}^1(p_2, \zeta e^{ip_1}) \rightarrow \mathcal{V}^1(p_1, \zeta e^{ip_2}) \otimes \mathcal{V}^1(p_2, \zeta),$$

where  $\mathcal{V}^1 \otimes \mathcal{V}^1 = \mathcal{W}^2$  is isomorphic to a typical (long) multiplet of dimension 16. Thus the  $S$ -matrix is described as the second-order differential operator

$$S^{AA} = \sum_{i=1}^{10} a_i \Lambda_i, \quad (4.1)$$

where the differential operators  $\Lambda_i$  are given in (4.5) of section 4.5 of [20]. The  $S$ -matrix  $S^{AA}$  coefficients  $a_i$  may be determined uniquely up to a overall constant using the symmetry algebra, and the full expression in our basis is given in the appendix. It was also shown that  $S^{AA}$  respects Yangian symmetry [19].

### **$S$ -matrix $S^{AB}$**

We define the  $S$ -matrix  $S^{AB}$  as an intertwining operator

$$S^{AB} : \mathcal{V}^1(p_1, \zeta) \otimes \mathcal{V}^2(p_2, \zeta e^{ip_1}) \rightarrow \mathcal{V}^1(p_1, \zeta e^{ip_2}) \otimes \mathcal{V}^2(p_2, \zeta),$$

where  $\mathcal{V}^1 \otimes \mathcal{V}^2 = \mathcal{W}^3$  is isomorphic to a long multiplet of dimension 32. Thus the  $S$ -matrix is described as the third-order differential operator

$$S^{AB} = \sum_{i=1}^{19} a_i \Lambda_i, \quad (4.2)$$

where the  $\Lambda_i$  are given in section 6.1.1 of [20]. The reflection coefficients  $a_i$  may be determined uniquely up to a overall constant using the symmetry algebra [20]<sup>3</sup>; the exact expression in our basis is again given in the appendix. It was also shown that  $S^{AB}$  respects Yangian symmetry [21, 22].

### **$S$ -matrix $S^{BB}$**

We define the  $S$ -matrix  $S^{BB}$  as an intertwining operator

$$S^{BB} : \mathcal{V}^2(p_1, \zeta) \otimes \mathcal{V}^2(p_2, \zeta e^{ip_1}) \rightarrow \mathcal{V}^2(p_1, \zeta e^{ip_2}) \otimes \mathcal{V}^2(p_2, \zeta),$$

where  $\mathcal{V}^2 \otimes \mathcal{V}^2 = \mathcal{W}^2 \oplus \mathcal{W}^4$  are long multiplets of dimension 16 and 48 respectively. Thus the  $S$ -matrix is described as the following fourth-order differential operator

$$S^{BB} = \sum_{i=1}^{48} a_i \Lambda_i, \quad (4.3)$$

with  $\Lambda_i$  as in section 6.2.1 of [20]. It was shown in [20] that the Lie superalgebra alone is not enough to fix all the coefficients  $a_i$ , and the YBE is required. This is the consequence of the decomposition of tensor product  $\mathcal{V}^2 \otimes \mathcal{V}^2$  being a sum of two long multiplets  $\mathcal{W}^2 \oplus \mathcal{W}^4$ . Therefore the  $S$ -matrix  $S^{BB}$  may be formally divided into two parts

$$S^{BB} = S_1^{BB} + q S_2^{BB}, \quad (4.4)$$

where  $q$  is a single parameter which is not determined by the symmetry algebra [20]. In the case of higher-order multi-magnon bound state  $S$ -matrices  $S^{MN}$ , there are precisely  $m - 1$  parameters that cannot be determined by the symmetry algebra alone, where  $m$  is the number of long multiplets in the decomposition of tensor product  $\mathcal{V}^M \otimes \mathcal{V}^{N4}$ . Hence, in

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<sup>3</sup>We found two typos in [20] in the coefficients of  $S^{AB}$  listed in 6.1.2. There should be a numerator  $(x_1^+ - y_2^+)$  instead of  $(x_1^- - y_2^+)$  in  $a_{13}$  and a numerator  $(1 - y_2^- x_1^+)$  instead of  $(1 - y_2^- x_1^-)$  in  $a_{14}$ . These typos were noted also in [35].

<sup>4</sup>The multiplet decomposition formula is given in p.19 of [20]. Long and short multiplets of  $\mathfrak{su}(2|2)$  have been studied in [5].

addition to the Lie superalgebra, the YBE or Yangian symmetry is required to find  $S^{MN}$  for  $M, N \geq 2$  [21, 22].

Precise expressions for the scattering coefficients  $a_1, \dots, a_{48}$  of the  $S$ -matrix  $S^{BB}$  were found in [20]<sup>5</sup>, by using the superalgebra together with YBE. It is interesting to note that  $a_{45}, \dots, a_{48}$  were found to be zero: the scattering channels  $\Lambda_{44}, \dots, \Lambda_{48}$  are forbidden, and  $a_{45} = \dots = a_{48} = 0$  independently of the choice of parametrization of  $a, b, c$  and  $d$ . In fact  $a_{45} = \dots = a_{48} = 0$  may be used as additional constraints and one can then obtain all the other scattering coefficients  $a_1, \dots, a_{44}$  using the symmetry algebra alone, a fact which we expect will be explained by a deeper understanding of the underlying Yangian symmetry.

The full expressions for the coefficients  $a_1, \dots, a_{48}$  of the  $S$ -matrix  $S^{BB}$  in our basis are again given in the appendix.

## 5. $K$ -matrices

### $K$ -matrix $K^{Aa}$

We define the  $K$ -matrix  $K^{Aa}$  as an intertwining operator

$$K^{Aa} : \mathcal{V}^1(p, \zeta) \otimes \mathcal{V}^1(q, \zeta e^{ip}) \rightarrow \mathcal{V}^1(-p, \zeta) \otimes \mathcal{V}^1(q, \zeta e^{-ip}),$$

corresponding to the tilded factor of the complete reflection  $K$ -matrix

$$\tilde{K} : \square \otimes \square \rightarrow \square \otimes \square.$$

The  $K$ -matrix  $K^{Aa}$  is given by the second-order differential operator

$$K^{Aa} = \sum_{i=1}^{10} k_i \Lambda_i, \tag{5.1}$$

with the  $\Lambda_i$  as in (4.5) of section 4.1 of [20]. The symmetry algebra fixes the values of the coefficients  $k_i$  uniquely up to a constant; again we reserve the full expression to the appendix. These coefficients were derived in [29].

### $K$ -matrix $K^{A1}$

We define the  $K$ -matrix  $K^{A1}$  as an intertwining operator

$$K^{Aa} : \mathcal{V}^1(p, \zeta) \otimes 1 \rightarrow \mathcal{V}^1(-p, \zeta) \otimes 1,$$

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<sup>5</sup>We found a typo in [20] in the coefficient  $a_{41}$  of  $S^{BB}$  listed in 6.2.2. There should be a numerator  $\tilde{\eta}_2$  instead of  $\tilde{\eta}_2^2$ . It is easy to see this by comparing the expressions of  $a_{41}$  with  $a_{42}$  and  $\Lambda_{41}$  with  $\Lambda_{42}$  from the section 6.2.1.

corresponding to the untilded factor of the complete reflection  $K$ -matrix

$$\tilde{K} : \square \otimes 1 \rightarrow \square \otimes 1,$$

The  $K$ -matrix  $K^{A1}$  may be expressed as a sum of diagonal first-order differential operators

$$K^{A1} = \sum_{i=1}^2 k_i \Lambda_i, \quad (5.2)$$

where

$$\Lambda_1 = \omega_a^1 \frac{\partial}{\partial \omega_a^1}, \quad \Lambda_2 = \theta_\alpha^1 \frac{\partial}{\partial \theta_\alpha^1}. \quad (5.3)$$

As was shown in [29], the symmetry algebra alone is not enough to fix  $k_1$  and  $k_2$ . Using the boundary Yang-Baxter equation (bYBE)

$$S^{AA}(p_1, p_2) K^{A1}(p_1) S^{AA}(p_2, -p_1) K^{A1}(p_2) = K^{A1}(p_2) S^{AA}(p_1, -p_2) K^{A1}(p_1) S^{AA}(p_1, p_2)$$

one finds that

$$\frac{k_2}{k_1} = \frac{x_B + x^+ \tilde{\eta}}{x_B - x^- \tilde{\eta}}. \quad (5.4)$$

The calculations are done as follows. First one must consider the matrix element

$$\langle e_1 \otimes e_3 | (\text{bYBE}) | e_1 \otimes e_3 \rangle, \quad (5.5)$$

where  $\langle e_1 \otimes e_3 |$  and  $| e_1 \otimes e_3 \rangle$  are the orthonormal vectors (3.19) from the superspaces  $\mathcal{V}^1(-p_1, \zeta) \otimes \mathcal{V}^1(-p_2, \zeta e^{-ip_1})$  and  $\mathcal{V}^1(p_1, \zeta) \otimes \mathcal{V}^1(p_2, \zeta e^{ip_1})$  respectively. This matrix element leads to the equation

$$\begin{aligned} & (k_1(p_2) \tilde{\eta}_2 - k_2(p_2) \eta_2) (k_2(p_1) \eta_1 x_1^- + k_1(p_1) \tilde{\eta}_1 x_1^+) \\ & - (k_1(p_1) \tilde{\eta}_1 - k_2(p_1) \eta_1) (k_2(p_2) \eta_2 x_2^- + k_1(p_2) \tilde{\eta}_2 x_2^+) = 0, \end{aligned} \quad (5.6)$$

which may be solved by separating variables and setting

$$\frac{k_2 \eta x^- + k_1 \tilde{\eta} x^+}{k_1 \tilde{\eta} - k_2 \eta} = x_B. \quad (5.7)$$

The solution of (5.7) gives the required equation (5.4). One must then show that the parameter  $x_B$  is indeed the spectral parameter of the boundary state. This may be achieved by considering the matrix element

$$\langle e_1 \otimes e_2 | (\text{bYBE}) | e_3 \otimes e_4 \rangle, \quad (5.8)$$

which may be set to zero only by requiring the parameter  $x_B$  to satisfy the mass-shell condition (3.17).



### **$K$ -matrix $K^{Ba}$**

We define the  $K$ -matrix  $K^{Ba}$  for reflection of a two-particle bound state  $B$  from the fundamental boundary state  $a$  as a third-order differential operator which acts as

$$K^{Ba}(p, q) : \quad \mathcal{V}^2(p, \zeta) \otimes \mathcal{V}^1(q, \zeta e^{ip}) \rightarrow \mathcal{V}^2(-p, \zeta) \otimes \mathcal{V}^1(q, \zeta e^{-ip}),$$

corresponding to the tilded factor of the complete reflection  $K$ -matrix

$$\tilde{K} : \boxed{\times} \otimes \boxed{\times} \rightarrow \boxed{\times} \otimes \boxed{\times}.$$

The  $K$ -matrix  $K^{Ba}$  is given by the following differential operator

$$K^{Ba} = \sum_{i=1}^{19} k_i \Lambda_i, \quad (5.9)$$

where the  $\Lambda_i$  may be easily obtained using the method described in section 3.2 of [20]. The symmetry algebra fixes the values of the coefficients  $k_i$  uniquely up to a constant; the full expression may be found in the appendix.

### **$K$ -matrix $K^{B1}$**

We define the  $K$ -matrix  $K^{B1}$  for reflection of a two-particle bound state  $B$  from the singlet boundary state 1 as a second-order differential operator which acts as

$$K^{B1}(p, q) : \quad \mathcal{V}^2(p, \zeta) \otimes 1 \rightarrow \mathcal{V}^2(-p, \zeta) \otimes 1,$$

corresponding to the untilded factor of the complete reflection  $K$ -matrix

$$K : \boxed{\times} \otimes 1 \rightarrow \boxed{\times} \otimes 1.$$

The  $K$ -matrix  $K^{B1}$  is given by the sum of diagonal differential operators

$$K^{B1} = \sum_{i=1}^3 k_i \Lambda_i, \quad (5.10)$$

where

$$\Lambda_1 = \omega_b^1 \omega_a^1 \frac{\partial^2}{\partial \omega_b^1 \partial \omega_a^1}, \quad \Lambda_2 = \omega_a^1 \theta_\alpha^1 \frac{\partial^2}{\partial \omega_a^1 \partial \theta_\alpha^1}, \quad \Lambda_3 = \theta_\beta^1 \theta_\alpha^1 \frac{\partial^2}{\partial \theta_\beta^1 \partial \theta_\alpha^1}. \quad (5.11)$$

Once again, the symmetry algebra alone is not enough to fix coefficients  $k_1$ ,  $k_2$  and  $k_3$ . We shall be using the bYBE

$$S^{AB}(p_1, p_2) K^{A1}(p_1) S^{AB}(p_1, -p_2) K^{B1}(p_2) = K^{B1}(p_2) S^{AB}(p_1, -p_2) K^{A1}(p_1) S^{AB}(p_1, p_2).$$

First we consider the matrix element

$$\langle e_3 \otimes e_{1,1} | (\text{bYBE}) | e_1 \otimes e_{1,3} \rangle. \quad (5.12)$$

Setting it to zero, we get a relation very similar to (5.4)

$$\frac{k_2}{k_1} = \frac{x_B + y^+}{x_B - y^-} \frac{\tilde{\eta}}{\eta}. \quad (5.13)$$

The second constraint is acquired by considering the matrix element

$$\langle e_3 \otimes e_{3,4} | (\text{bYBE}) | e_1 \otimes e_{2,3} \rangle, \quad (5.14)$$

which gives the ratio

$$\frac{k_3}{k_2} = \frac{1 - x_B y^+}{1 + x_B y^-} \frac{\tilde{\eta}}{\eta}. \quad (5.15)$$

Using the convenient normalization  $k_1 = 1$ , we obtain

$$\begin{aligned} k_1 &= 1, \\ k_2 &= \frac{x_B + y^+}{x_B - y^-} \frac{\tilde{\eta}}{\eta}, \\ k_3 &= \frac{(x_B + y^+)(1 - x_B y^+)}{(x_B - y^-)(1 + x_B y^-)} \frac{\tilde{\eta}^2}{\eta^2}. \end{aligned} \quad (5.16)$$

The reflection matrix  $K^{B1}$  naturally extends to the reflection of any bound state  $K^{M1}$  with  $M \geq 2$  as there are always only three diagonal reflection coefficients with the following differential operators

$$\begin{aligned} \Lambda_1 &= \omega_{a_M}^1 \cdots \omega_{a_1}^1 \frac{\partial^M}{\partial \omega_{a_M}^1 \cdots \partial \omega_{a_1}^1}, \\ \Lambda_2 &= \omega_{a_{M-1}}^1 \cdots \omega_{a_1}^1 \theta_\alpha^1 \frac{\partial^M}{\partial \omega_{a_{M-1}}^1 \cdots \partial \omega_{a_1}^1 \partial \theta_\alpha^1}, \\ \Lambda_3 &= \omega_{a_{M-2}}^1 \cdots \omega_{a_1}^1 \theta_\beta^1 \theta_\alpha^1 \frac{\partial^M}{\partial \omega_{a_{M-2}}^1 \cdots \partial \omega_{a_1}^1 \partial \theta_\beta^1 \partial \theta_\alpha^1}, \end{aligned} \quad (5.17)$$

and the invariance under bYBE leads precisely to the same reflection coefficients as in (5.16) subject to the mass shell conditions (3.12) and (3.17).

### **$K$ -matrix $K^{Ab}$**

We define the  $K$ -matrix  $K^{Ab}$  for reflection of a fundamental bulk state  $A$  from a two-particle bound state  $b$  on the boundary as a third-order differential operator which acts as

$$K^{Ab}(p, q) : \mathcal{V}^1(p, \zeta) \otimes \mathcal{V}^2(q, \zeta e^{ip}) \rightarrow \mathcal{V}^1(-p, \zeta) \otimes \mathcal{V}^2(q, \zeta e^{-ip}).$$

corresponding to the tilded factor of the complete reflection  $K$ -matrix

$$\tilde{K} : \square \otimes \square \square \rightarrow \square \otimes \square \square.$$

The  $K$ -matrix  $K^{Ab}$  is given by the differential operator

$$K^{Ab} = \sum_{i=1}^{19} k_i \Lambda_i, \quad (5.18)$$

where the  $\Lambda_i$  are as in section 6.1.1 of [20]. The symmetry algebra fixes the values of the coefficients  $k_i$  uniquely up to a constant; again the full expression may be found in the appendix.

### **$K$ -matrix $K^{Bb}$**

We define the  $K$ -matrix  $K^{Bb}$  for reflection of a bulk two-magnon bound state  $B$  from a two-magnon bound state  $b$  on the boundary as a fourth-order differential operator which acts as

$$K^{Bb}(p, q) : \mathcal{V}^2(p, \zeta) \otimes \mathcal{V}^2(q, \zeta e^{ip}) \rightarrow \mathcal{V}^2(-p, \zeta) \otimes \mathcal{V}^2(q, \zeta e^{-ip}).$$

corresponding to the tilded factor of the complete reflection  $K$ -matrix

$$\tilde{K} : \boxtimes \boxtimes \otimes \boxtimes \boxtimes \rightarrow \boxtimes \boxtimes \otimes \boxtimes \boxtimes.$$

As a differential operator the  $K$ -matrix is

$$K^{Ab} = \sum_{i=1}^{48} k_i \Lambda_i, \quad (5.19)$$

where the  $\Lambda_i$  are as given in section 6.2.1 of [20]. By choosing the constraints to be  $k_1 = 1$  and  $k_{45}, \dots, k_{48} = 0$ , in the same fashion as we did for the bulk case, we were able to find the expressions for  $k_2, \dots, k_{44}$  using the symmetry algebra alone. The values of the coefficients  $k_i$  may be found in the appendix.

## **6. Conclusions and Outlook**

In this paper we have used the superspace formalism introduced in [20] to obtain the reflection matrices of magnon bound states from the  $Z = 0$  D7 brane [29]. The matter fields in the bulk transform in representations of  $\mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2)$ , while on the boundary the residual Lie symmetry, preserved by the reflection of bulk excitations, is  $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \widetilde{\mathfrak{psu}}(2|2)$ . The reflection matrix factors as a tensor product for tilded and untilded factors. The reflection of fundamental magnons was rigorously worked out in [29].

Here we have calculated the scattering matrices  $K^{Ba}$ ,  $K^{Ab}$  and  $K^{Bb}$  describing the reflections of two-magnon bound states in the tilded factor. We found that  $K^{Ba}$  and  $K^{Ab}$  may be determined up to an overall factor by the symmetry algebra alone, just as for the bulk  $S^{BA}$  of [20]. However, as for the bulk  $S^{BB}$ , we found that  $K^{Bb}$  is not determined uniquely by the symmetry algebra, but must be constrained by the bYBE. Alternatively,

it turns out that we may set the coefficients  $k_{45}, \dots, k_{48}$  to zero and then use the symmetry algebra alone.

Further, we have calculated the reflection matrix  $K^{B1}$  describing the reflection in the untilded factor of the two-magnon bound state in the bulk from a singlet boundary state. This may naturally be generalized to any  $K^{M1}$  with  $M \geq 2$ , as these have only three reflection coefficients, which may be fixed up to an overall scalar factor by requiring that the bYBE be satisfied. This is in agreement with the results of [29] for the reflection of fundamental states.

An important question is of the physicality of the boundary bound states. The bound state  $K$ -matrices we have constructed here have a pole (and zero) structure very similar to the bulk case, so that the pole  $x^+ = x_B$  appearing throughout the  $K$ -matrices signals the presence of higher order multi-magnon bound states. As was noted in [33], an open string ending on a giant magnon is expected to have a tower of multi-magnon bound states on the boundary for any value of the coupling constant  $g$ . The tilded factor of the  $Z = 0$  D7 brane shares the same symmetry as the  $Z = 0$  giant magnon; thus we expect multi-magnon bound states to be living on the tilded factor of our boundary [34]. But this is not the case for the untilded factor. We do not expect the pole in the reflection matrix  $K^{M1}$  to correspond to a physical bound state for any  $M$ , since the values of  $\Delta - J_{56}$  on the boundary fields (and thereby states) [29] are inconsistent with the pole residues, at least for the low-lying states. However, the issue of the physicality of putative bound states cannot be fully resolved without an analysis of the boundary on-shell (Landau) diagrams [32, 43, 44], which is beyond the scope of the present work.

The problem remains of understanding the structure of the scattering matrices we have found. One would expect that, where the bYBE was needed in addition to the Lie superalgebra, the underlying Yangian symmetry should supply the deficit, as in the bulk case [21, 22]. In the boundary case we would expect to see some form of generalized twisted Yangian [45] as the boundary remnant of the bulk Yangian symmetry, and that this would, for example, be sufficient to fix  $K^{B1}$  up to an overall factor, as happens in the maximal giant graviton (D3) case [35]. Similarly for  $K^{Bb}$  we would hope that the Yangian symmetry would explain the curious zeros in  $k_{45}, \dots, k_{48}$ , and organize the coefficients of  $K^{Bb}$  more sensibly. This is the subject of current investigations.

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## A. Appendices

### A.1 Coefficients of $S$ -matrices

We use the convenient notation of the bulk spectral parameters, where  $x_1^\pm, x_2^\pm$  are the parameters of the fundamental states with momentum  $p_1$  and  $p_2$ , while  $y_1^\pm, y_2^\pm$  are the parameters of the two-particle bound states respectively with momentum  $p_1$  and  $p_2$ . The parameters  $\eta_i$  are given by

$$\eta_1 = \eta(p_1), \quad \eta_2 = e^{\frac{i}{2}p_1} \eta(p_2), \quad \tilde{\eta}_1 = e^{\frac{i}{2}p_2} \eta(p_1), \quad \tilde{\eta}_2 = \eta(p_2), \quad (\text{A.1})$$

where  $\eta(p_i) = e^{i\xi_0} e^{\frac{i}{4}p_i} \sqrt{i(x_i^- - x_i^+)}$  for the fundamental states and  $x_i^\pm$  must be changed into  $y_i^\pm$  for the bound states. The rapidity parameters  $u_i$  used in the expressions for the coefficients of the  $S$ -matrix  $S^{BB}$  are expressed in terms of  $y_i^\pm$  as follows

$$u_i = \frac{1}{2} \left( y_i^+ + \frac{1}{y_i^+} + y_i^- + \frac{1}{y_i^-} \right). \quad (\text{A.2})$$

#### The $S$ -matrix $S^{AA}$

The  $S$ -matrix  $S^{AA}$  has the following components  $a_i$ :

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 2 \frac{x_2^+ (x_1^- - x_2^-) (-1 + x_1^+ x_2^-)}{x_2^- (x_1^- - x_2^+) (-1 + x_1^+ x_2^+)} - 1, \\ a_3 &= - \frac{(x_2^- - x_1^+) \tilde{\eta}_1 \tilde{\eta}_2}{(x_1^- - x_2^+) \eta_1 \eta_2}, \\ a_4 &= \left( \frac{x_2^- - x_1^+}{x_1^- - x_2^+} + 2 \frac{x_1^+ (x_1^- - x_2^-) (-1 + x_1^- x_2^+)}{x_1^- (x_1^- - x_2^+) (-1 + x_1^+ x_2^+)} \right) \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2}, \\ a_5 &= \frac{(x_1^+ - x_2^+) \tilde{\eta}_2}{(x_1^- - x_2^+) \eta_1}, \\ a_6 &= \frac{(x_1^- - x_2^-) \tilde{\eta}_1}{(x_1^- - x_2^+) \eta_1}, \\ a_7 &= - \frac{i \zeta x_1^+ (x_1^- - x_2^-) (x_1^- - x_1^+) (x_2^- - x_2^+)}{x_1^- x_2^- (x_1^- - x_2^+) (-1 + x_1^+ x_2^+) \eta_1 \eta_2}, \\ a_8 &= \frac{i (x_1^- - x_2^-) \tilde{\eta}_1 \tilde{\eta}_2}{\zeta (x_1^- - x_2^+) (-1 + x_1^+ x_2^+)}, \\ a_9 &= \frac{(x_1^- - x_1^+) \tilde{\eta}_2}{(x_1^- - x_2^+) \eta_1}, \\ a_{10} &= \frac{(x_2^- - x_2^+) \tilde{\eta}_1}{(x_1^- - x_2^+) \eta_2}. \end{aligned}$$

These coefficients are in agreement with the ones found in [8] up to a relative sign corresponding to the eigenvalue of the graded permutation operator acting on the antisymmetric states (see footnote on page 11) and an overall factor

$$N_0^{AA} = \frac{x_1^- - x_2^+}{x_2^- - x_1^+}.$$

This corresponds to the normalization  $a_3 = 1$ , which we might refer to as the ‘physical normalization’ since it removes the pole in the  $S$ -matrix element  $a_3$ , which would produce a state symmetric in fermionic indices state and therefore cannot yield a bound state. Rather it is the pole at  $x_2^- - x_1^+$  which is responsible for the creation of bound states, although this pole is typically hidden in the normalization  $a_1 = 1$  used in calculations.

### The $S$ -matrix $S^{AB}$

The  $S$ -matrix  $S^{AB}$  has the following components  $a_i$ :

$$\begin{aligned} a_1 &= 1, \\ a_2 &= -\frac{1}{2} + \frac{3y_2^+ (x_1^- - y_2^-) (-1 + x_1^+ y_2^-)}{2y_2^- (x_1^- - y_2^+) (-1 + x_1^+ y_2^+)}, \\ a_3 &= \frac{(x_1^+ - y_2^+) \tilde{\eta}_2}{(x_1^- - y_2^+) \eta_2}, \\ a_4 &= \frac{y_2^+ (x_1^+ - y_2^-) (-1 + x_1^+ y_2^-) \tilde{\eta}_2}{y_2^- (x_1^- - y_2^+) (-1 + x_1^+ y_2^+) \eta_2}, \\ a_5 &= \frac{(x_1^- - y_2^-) \tilde{\eta}_1}{(x_1^- - y_2^+) \eta_1}, \\ a_6 &= \frac{x_1^+ (x_1^+ y_2^- + x_1^+ y_2^+ - 2y_2^- y_2^+ - 2x_1^- x_1^+ y_2^- y_2^+ + x_1^- (y_2^-)^2 y_2^+ + x_1^- y_2^- (y_2^+)^2) \tilde{\eta}_2^2}{2x_1^- y_2^- (x_1^- - y_2^+) (1 - x_1^+ y_2^+) \eta_2^2}, \\ a_7 &= -\frac{(x_1^+ - y_2^-) \tilde{\eta}_1 \tilde{\eta}_2}{(x_1^- - y_2^+) \eta_1 \eta_2}, \\ a_8 &= -\left( \frac{2x_1^+ (x_1^- - y_2^-) (-1 + x_1^- y_2^+)}{x_1^- (x_1^- - y_2^+) (-1 + x_1^+ y_2^+)} - \frac{x_1^+ - y_2^-}{x_1^- - y_2^+} \right) \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2}, \\ a_9 &= \frac{x_1^+ (x_1^+ - y_2^-) (-1 + x_1^- y_2^+) \tilde{\eta}_1 \tilde{\eta}_2^2}{x_1^- (x_1^- - y_2^+) (-1 + x_1^+ y_2^+) \eta_1 \eta_2^2}, \\ a_{10} &= -\frac{i\zeta (x_1^- - x_1^+) x_1^+ (y_2^- - y_2^+)^2 y_2^+}{2x_1^- y_2^- (x_1^- - y_2^+) (-1 + x_1^+ y_2^+) \eta_2^2}, \\ a_{11} &= \frac{i (x_1^- - x_1^+) \tilde{\eta}_2^2}{2\zeta (x_1^- - y_2^+) (-1 + x_1^+ y_2^+)}, \\ a_{12} &= \frac{i\zeta x_1^+ y_2^+ (x_1^- - x_1^+) (x_1^- - y_2^-) (y_2^- - y_2^+)}{\sqrt{2} x_1^- y_2^- (-x_1^- + y_2^+) (-1 + x_1^+ y_2^+) \eta_1 \eta_2}, \end{aligned}$$

$$\begin{aligned}
a_{13} &= -\frac{i(x_1^- - y_2^-)\tilde{\eta}_1\tilde{\eta}_2}{\sqrt{2}\zeta(x_1^- - y_2^+)(-1 + x_1^+y_2^+)}, \\
a_{14} &= \frac{(x_1^- - x_1^+)x_1^+(-1 + x_1^-y_2^+)\tilde{\eta}_2^2}{\sqrt{2}x_1^-(x_1^- - y_2^+)(-1 + x_1^+y_2^+)\eta_1\eta_2}, \\
a_{15} &= \frac{x_1^+(-y_2^- + y_2^+)(-1 + x_1^-y_2^+)\tilde{\eta}_1\tilde{\eta}_2}{\sqrt{2}x_1^-(x_1^- - y_2^+)(-1 + x_1^+y_2^+)\eta_2^2}, \\
a_{16} &= \frac{i(x_1^+ - y_2^-)\tilde{\eta}_1\tilde{\eta}_2^2}{\sqrt{2}\zeta(x_1^- - y_2^+)(-1 + x_1^+y_2^+)\eta_2}, \\
a_{17} &= \frac{i\zeta x_1^+y_2^+(x_1^- - x_1^+)(x_1^+ - y_2^-)(y_2^- - y_2^+)\tilde{\eta}_2}{\sqrt{2}x_1^-y_2^-(x_1^- + y_2^+)(-1 + x_1^+y_2^+)\eta_1\eta_2^2}, \\
a_{18} &= \frac{(x_1^- - x_1^+)\tilde{\eta}_2}{\sqrt{2}(x_1^- - y_2^+)\eta_1}, \\
a_{19} &= \frac{(y_2^- - y_2^+)\tilde{\eta}_1}{\sqrt{2}(x_1^- - y_2^+)\eta_2}.
\end{aligned}$$

### The $S$ -matrix $S^{BB}$

The  $S$ -matrix  $S^{BB}$  has the following components  $a_i$ :

$$\begin{aligned}
a_1 &= 1, \\
a_2 &= \frac{(y_2^- - y_1^+)(-1 + y_2^-y_1^+)y_2^+}{(-1 + y_1^-y_2^-)y_1^+(y_1^- - y_2^+)} \\
&\quad \times \frac{(y_1^-(2 + y_2^-(y_1^- - 3y_1^+))y_1^+ + (y_1^+ + y_1^-(3 + 2y_2^-y_1^+))y_2^+)}{(-3y_2^-y_1^+ + (y_1^+ + y_2^-(2 + y_1^+(y_2^- + 2y_1^+)))y_2^+ - 3y_2^-y_1^+(y_2^+)^2)}, \\
a_3 &= \frac{1}{2y_2^-( -1 + y_1^-y_2^-)(y_1^+)^2(y_1^- - y_2^+)y_2^+\left(u_1 - u_2 - \frac{2i}{g}\right)} \\
&\quad \times \left( (y_1^-)^2(y_2^-)^2(y_1^+)^2 + (y_1^+)^2\left(y_2^-(2 + (y_2^- - y_2^+)^2) - y_2^+\right)y_2^+ \right. \\
&\quad \quad + y_1^-y_2^-\left(2y_2^-(y_1^+)^4y_2^+ + 2(y_2^+)^2 - y_1^+y_2^+\left(5 + (y_2^-)^2 + y_2^-y_2^+ + 2(y_2^+)^2\right) \right. \\
&\quad \quad \quad \left. + (y_1^+)^2\left(2 + (y_2^-)^2 - y_2^-\left(-1 + (y_2^-)^2\right)y_2^+ + 3\left(2 + (y_2^-)^2\right)(y_2^+)^2\right) \right. \\
&\quad \quad \quad \left. \left. - (y_1^+)^3(3y_2^+ + y_2^-(2 + y_2^+(y_2^- + 3y_2^+)))\right) \right), \\
a_4 &= -\frac{(y_2^- - y_1^+)\tilde{\eta}_1\tilde{\eta}_2}{(y_1^- - y_2^+)\eta_1\eta_2}, \\
a_5 &= -\left(\frac{2(y_1^- - y_2^-)(-y_1^+ + y_2^+)(-1 + y_1^-y_2^+)}{y_1^-(y_1^- - y_2^+)y_2^+\left(u_1 - u_2 - \frac{2i}{g}\right)} - \frac{(y_2^- - y_1^+)}{(y_1^- - y_2^+)}\right)\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1\eta_2},
\end{aligned}$$

$$\begin{aligned}
a_6 &= -\frac{(y_2^- - y_1^+)}{(y_1^- - y_2^+)} \frac{(u_1 - u_2 + \frac{2i}{g})}{(u_1 - u_2 - \frac{2i}{g})} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2}, \\
a_7 &= -\frac{(y_2^- - y_1^+)}{2y_2^- (-1 + y_1^- y_2^-) y_1^+ (y_1^- - y_2^+) y_2^+ \left(u_1 - u_2 - \frac{2i}{g}\right)} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1 \eta_2} \\
&\quad \times \left( 2y_2^- y_1^+ y_2^+ \left(u_1 - u_2 + \frac{2i}{g}\right) + y_2^- \left(-2(2 + y_1^- (y_2^- - 2y_2^+)) y_2^+ \right. \right. \\
&\quad \left. \left. + 2y_2^- (y_1^+)^2 (-2 + y_1^- y_2^+) + y_1^+ (4 + y_1^- (y_2^- - 3y_2^+)) (1 + y_2^- y_2^+) \right) \right), \\
a_8 &= \frac{(y_2^- - y_1^+) (y_1^- y_2^- - 2y_1^- y_1^+ + y_2^- y_1^+) (-1 + y_1^- y_2^+)}{2 (y_1^-)^2 y_2^+ (-y_1^- + y_2^+) (-1 + y_1^+ y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right)} \frac{\tilde{\eta}_1^2 \tilde{\eta}_2^2}{\eta_1^2 \eta_2^2} \\
&\quad - \frac{(y_2^- - y_1^+) y_1^+ (y_1^- - 2y_2^- + y_1^+) (-1 + y_1^- y_2^+)}{2y_1^- (y_1^- - y_2^+) (-1 + y_1^+ y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right)} \frac{\tilde{\eta}_1^2 \tilde{\eta}_2^2}{\eta_1^2 \eta_2^2}, \\
a_9 &= \frac{(y_1^+ - y_2^+)}{(y_1^- - y_2^+)} \frac{\tilde{\eta}_2}{\eta_2}, \\
a_{10} &= \frac{(y_1^- - y_2^-)}{(y_1^- - y_2^+)} \frac{\tilde{\eta}_1}{\eta_1}, \\
a_{11} &= -\frac{(y_2^- - y_1^+) (y_1^+ - y_2^+) (-1 + y_2^- y_1^+)}{y_2^- y_1^+ (y_1^- - y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right)} \frac{\tilde{\eta}_2}{\eta_2}, \\
a_{12} &= \frac{(-y_1^- + y_2^-) (y_2^- - y_1^+) (-1 + y_2^- y_1^+)}{y_2^- y_1^+ (y_1^- - y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right)} \frac{\tilde{\eta}_1}{\eta_1}, \\
a_{13} &= \frac{(y_1^+ - y_2^+) \tilde{\eta}_2^2}{2y_1^- y_2^- (-1 + y_1^- y_2^-) y_1^+ (y_1^- - y_2^+) y_2^+ \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_2^2}, \\
&\quad \times \left( y_1^+ \left( y_2^- y_1^+ + \left( y_1^+ + y_2^- \left( -2 + (y_2^-)^2 - 2y_2^- y_1^+ \right) \right) y_2^+ + (y_2^-)^2 (y_2^+)^2 \right) \right. \\
&\quad \left. + y_1^- y_2^- \left( - (y_2^-)^3 y_1^+ y_2^+ - (y_1^+)^2 y_2^+ + (y_2^-)^2 \left( y_1^+ + \left( -1 + (y_1^+)^2 \right) y_2^+ \right) \right. \right. \\
&\quad \left. \left. + y_2^- \left( 2 (y_1^+)^3 y_2^+ - (y_2^+)^2 - 3 (y_1^+)^2 \left( 1 + (y_2^+)^2 \right) + y_1^+ y_2^+ \left( 5 + (y_2^+)^2 \right) \right) \right) \right), \\
a_{14} &= \left( \frac{(y_1^- - y_2^-) (y_1^- - 2y_2^- + y_1^+)}{2 (y_1^- - y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right)} + \frac{(y_1^- - y_2^-) (y_1^- (y_2^- - 2y_1^+) + y_2^- y_1^+)}{2y_1^- y_1^+ (y_1^- - y_2^+) y_2^+ \left(u_1 - u_2 - \frac{2i}{g}\right)} \right) \frac{\tilde{\eta}_1^2}{\eta_1^2}, \\
a_{15} &= -\frac{(y_1^- - y_2^-) (y_2^- - y_1^+) (-1 + y_1^- y_2^+)}{y_2^+ \left( (y_1^-)^2 - y_1^- y_2^+ \right) \left(u_1 - u_2 - \frac{2i}{g}\right)} \frac{\tilde{\eta}_1^2 \tilde{\eta}_2}{\eta_1^2 \eta_2}, \\
a_{16} &= \frac{(y_2^- - y_1^+) (y_1^+ - y_2^+) (-1 + y_1^- y_2^+)}{y_1^- \left( -y_1^- y_2^+ + (y_2^+)^2 \right) \left(u_1 - u_2 - \frac{2i}{g}\right)} \frac{\tilde{\eta}_1 \tilde{\eta}_2^2}{\eta_1 \eta_2^2},
\end{aligned}$$



$$\begin{aligned}
a_{17} &= -\frac{\zeta^2 (y_1^- - y_2^-) (y_1^- - y_1^+)^2 (y_2^- - y_1^+) y_1^+ (y_2^- - y_2^+)^2 y_2^+}{2 (y_1^-)^2 (y_2^-)^2 (y_1^- - y_2^+) (-1 + y_1^+ y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1^2 \eta_2^2}, \\
a_{18} &= -\frac{y_1^- y_2^- (y_2^- - y_1^+) (y_1^+ - y_2^+) \tilde{\eta}_1^2 \tilde{\eta}_2^2}{2\zeta^2 (-1 + y_1^- y_2^-) (y_1^+)^2 (y_1^- - y_2^+) (y_2^+)^2 \left(u_1 - u_2 - \frac{2i}{g}\right)}, \\
a_{19} &= \frac{i\zeta (y_1^- - y_1^+) (y_2^- - y_1^+) (-1 + y_2^- y_1^+) (y_2^- - y_2^+) (y_1^+ - y_2^+)}{2y_2^- (-1 + y_1^- y_2^-) y_1^+ (y_1^- - y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1 \eta_2}, \\
a_{20} &= -\frac{i y_1^- (y_2^- - y_1^+) (-1 + y_2^- y_1^+) (y_1^+ - y_2^+) \tilde{\eta}_1 \tilde{\eta}_2}{2\zeta (-1 + y_1^- y_2^-) (y_1^+)^2 (y_1^- - y_2^+) y_2^+ \left(u_1 - u_2 - \frac{2i}{g}\right)}, \\
a_{21} &= \frac{i y_2^- (y_2^- - y_1^+) (y_1^+ - y_2^+) (-1 + y_1^- y_2^+) \tilde{\eta}_1^2 \tilde{\eta}_2^2}{2\zeta (-1 + y_1^- y_2^-) y_1^+ (y_1^- - y_2^+) (y_2^+)^2 \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1 \eta_2}, \\
a_{22} &= -\frac{i\zeta (y_1^- - y_2^-) (y_1^- - y_1^+) (y_2^- - y_1^+) y_1^+ (y_2^- - y_2^+) (-1 + y_1^- y_2^+) \tilde{\eta}_1 \tilde{\eta}_2}{2 (y_1^-)^2 y_2^- (y_1^- - y_2^+) (-1 + y_1^+ y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1^2 \eta_2^2}, \\
a_{23} &= \frac{i\zeta (y_1^- - y_1^+) (y_2^- - y_2^+) (y_1^+ - y_2^+)^2 (-1 + y_1^+ y_2^+)}{2 (-1 + y_1^- y_2^-) y_1^+ y_2^+ (-y_1^- + y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1 \eta_2}, \\
a_{24} &= \frac{i y_1^- y_2^- (y_1^+ - y_2^+)^2 (-1 + y_1^+ y_2^+) \tilde{\eta}_1 \tilde{\eta}_2}{2\zeta (-1 + y_1^- y_2^-) (y_1^+)^2 (y_1^- - y_2^+) (y_2^+)^2 \left(u_1 - u_2 - \frac{2i}{g}\right)}, \\
a_{25} &= \frac{(y_1^- - y_1^+)^2 (-1 + y_1^- y_2^+) \tilde{\eta}_2^2}{2y_1^- (y_1^- - y_2^+) y_2^+ \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1^2}, \\
a_{26} &= \frac{(y_2^- - y_2^+)^2 (-1 + y_1^- y_2^+) \tilde{\eta}_1^2}{2y_1^- (y_1^- - y_2^+) y_2^+ \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_2^2}, \\
a_{27} &= \frac{(y_1^- - y_1^+) (-y_2^- + y_1^+) (-1 + y_2^- y_1^+) \tilde{\eta}_2}{2y_2^- y_1^+ (y_1^- - y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1}, \\
a_{28} &= -\frac{(y_2^- - y_1^+) (-1 + y_2^- y_1^+) (y_2^- - y_2^+) \tilde{\eta}_1}{2y_2^- y_1^+ (y_1^- - y_2^+) \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_2}, \\
a_{29} &= \frac{(y_1^- - y_1^+) (-y_2^- + y_1^+) (-1 + y_1^- y_2^+) \tilde{\eta}_1 \tilde{\eta}_2^2}{2y_2^+ \left((y_1^-)^2 - y_1^- y_2^+\right) \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1^2 \eta_2}, \\
a_{30} &= \frac{(y_2^- - y_1^+) (y_2^- - y_2^+) (-1 + y_1^- y_2^+) \tilde{\eta}_1^2 \tilde{\eta}_2}{2y_1^- \left(-y_1^- y_2^+ + (y_2^+)^2\right) \left(u_1 - u_2 - \frac{2i}{g}\right) \eta_1 \eta_2^2}, \\
a_{31} &= \frac{(y_1^- - y_1^+) \tilde{\eta}_2}{(y_1^- - y_2^+) \eta_1}, \\
a_{32} &= \frac{(y_2^- - y_2^+) \tilde{\eta}_1}{(y_1^- - y_2^+) \eta_2},
\end{aligned}$$

$$\begin{aligned}
a_{33} &= \frac{(y_1^- - y_1^+)(y_1^+ - y_2^+)(-1 + y_1^- y_2^+)}{2y_2^+ \left( (y_1^-)^2 - y_1^- y_2^+ \right) \left( u_1 - u_2 - \frac{2i}{g} \right)} \frac{\tilde{\eta}_2^2}{\eta_1 \eta_2}, \\
a_{34} &= -\frac{(y_2^- - y_2^+)(-y_1^+ + y_2^+)(-1 + y_1^- y_2^+)}{2y_1^- (y_1^- - y_2^+) y_2^+ \left( u_1 - u_2 - \frac{2i}{g} \right)} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_2^2}, \\
a_{35} &= \frac{(y_1^- - y_2^-)(y_2^- - y_2^+)(-1 + y_1^- y_2^+)}{2y_2^+ \left( (y_1^-)^2 - y_1^- y_2^+ \right) \left( u_1 - u_2 - \frac{2i}{g} \right)} \frac{\tilde{\eta}_1^2}{\eta_1 \eta_2}, \\
a_{36} &= \frac{(y_1^- - y_2^-)(y_1^- - y_1^+)(-1 + y_1^- y_2^+)}{2y_2^+ \left( (y_1^-)^2 - y_1^- y_2^+ \right) \left( u_1 - u_2 - \frac{2i}{g} \right)} \frac{\tilde{\eta}_1 \tilde{\eta}_2}{\eta_1^2}, \\
a_{37} &= -\frac{i\zeta (y_1^- - y_1^+)(y_2^- - y_2^+)^2 (y_1^+ - y_2^+)}{2y_1^- y_2^- (y_1^- - y_2^+) \left( u_1 - u_2 - \frac{2i}{g} \right)} \eta_2^2, \\
a_{38} &= \frac{i (y_1^- - y_1^+)(y_1^+ - y_2^+) \tilde{\eta}_2^2}{2\zeta y_1^+ (y_1^- - y_2^+) y_2^+ \left( u_1 - u_2 - \frac{2i}{g} \right)}, \\
a_{39} &= -\frac{i\zeta (y_1^- - y_2^-)(y_1^- - y_1^+)^2 (y_2^- - y_2^+)}{2y_1^- y_2^- (y_1^- - y_2^+) \left( u_1 - u_2 - \frac{2i}{g} \right)} \eta_1^2, \\
a_{40} &= \frac{i (y_1^- - y_2^-)(y_2^- - y_2^+) \tilde{\eta}_1^2}{2\zeta y_1^+ (y_1^- - y_2^+) y_2^+ \left( u_1 - u_2 - \frac{2i}{g} \right)}, \\
a_{41} &= \frac{i\zeta (y_1^- - y_1^+)(y_2^- - y_1^+)(y_2^- - y_2^+)(y_1^+ - y_2^+)}{2y_1^- y_2^- (y_1^- - y_2^+) \left( u_1 - u_2 - \frac{2i}{g} \right)} \frac{\tilde{\eta}_2}{\eta_1 \eta_2^2}, \\
a_{42} &= -\frac{i (y_2^- - y_1^+)(y_1^+ - y_2^+)}{2\zeta y_1^+ (y_1^- - y_2^+) y_2^+ \left( u_1 - u_2 - \frac{2i}{g} \right)} \frac{\tilde{\eta}_1 \tilde{\eta}_2^2}{\eta_2}, \\
a_{43} &= \frac{i\zeta (y_1^- - y_2^-)(y_1^- - y_1^+)(y_2^- - y_1^+)(y_2^- - y_2^+)}{2y_1^- y_2^- (y_1^- - y_2^+) \left( u_1 - u_2 - \frac{2i}{g} \right)} \frac{\tilde{\eta}_1}{\eta_1^2 \eta_2}, \\
a_{44} &= -\frac{i (y_1^- - y_2^-)(y_2^- - y_1^+)}{2\zeta y_1^+ (y_1^- - y_2^+) y_2^+ \left( u_1 - u_2 - \frac{2i}{g} \right)} \frac{\tilde{\eta}_1^2 \tilde{\eta}_2}{\eta_1}, \\
a_{45} &= a_{46} = a_{47} = a_{48} = 0.
\end{aligned}$$

## A.2 Coefficients of $K$ -matrices

We use the convenient notation for the spectral parameters in which  $x^\pm$  and  $y^\pm$  are the spectral parameters of the fundamental state and two-particle bound state (respectively) in the bulk with momentum  $p$ , while  $x_B$  and  $y_B$  are the spectral parameters of the fundamental state and two-particle bound state respectively on the boundary. The bulk parameters change as  $x^\pm \rightarrow -x^\mp$ ,  $y^\pm \rightarrow -y^\mp$  and  $\eta \rightarrow \tilde{\eta}$  under the reflection. The boundary parameter  $\eta_B$  is related to the boundary spectral parameters as  $|\eta_B|^2 = -ix_B$

and  $|\eta_B|^2 = -iy_B$  in the cases of fundamental and two-particle bound states respectively and changes to  $\tilde{\eta}_B$  under the reflection.

### The $K$ -matrix $K^{Aa}$

The  $K$ -matrix  $K^{Aa}$  has the following components  $k_i$ :

$$\begin{aligned}
k_1 &= 1, \\
k_2 &= 1 + 2 \frac{(x_B + x^-) \left( (x^-)^2 - (x^+)^2 \right)}{(x_B - x^-) x^- x^+}, \\
k_3 &= - \frac{x^+ (x_B + x^+) \tilde{\eta} \tilde{\eta}_B}{(x_B - x^-) x^- \eta \eta_B}, \\
k_4 &= \left( \frac{(x^- - 2x^+) (x_B + x^- - x^+) (x^- + x^+)}{(x_B - x^-) (x^-)^2} + 1 \right) \frac{\tilde{\eta} \tilde{\eta}_B}{\eta \eta_B}, \\
k_5 &= \frac{(x_B x^- - (x^+)^2) \tilde{\eta}_B}{(x_B - x^-) x^- \eta_B}, \\
k_6 &= - \frac{\left( (x^-)^2 + x_B x^+ \right) \tilde{\eta}}{(x_B - x^-) x^- \eta}, \\
k_7 &= - \frac{i \zeta x_B (x_B + x^- - x^+) \left( (x^-)^2 - (x^+)^2 \right)}{(x_B - x^-) x^- \eta \eta_B}, \\
k_8 &= \frac{i (x_B + x^- - x^+) (x^- + x^+) \tilde{\eta} \eta_B}{\zeta (x_B - x^-) x^-}, \\
k_9 &= \frac{\left( -(x^-)^2 + (x^+)^2 \right) \tilde{\eta}_B}{(x_B - x^-) x^- \eta}, \\
k_{10} &= \frac{x_B (x^- + x^+) \tilde{\eta}}{(x_B - x^-) x^- \eta_B}.
\end{aligned}$$

Our coefficients are in agreement with the ones found in [29] up to an overall factor

$$N_0^{Aa} = \frac{x^- (x^- - x_B)}{x^+ (x^+ + x_B)},$$

which corresponds to normalization with  $k_3 = 1$  that once again may be called as a physical normalization, because in the same way as for  $S$ -matrix, the  $K$ -matrix element  $k_3$  shouldn't have a pole as it produces a symmetric in fermionic indices state which can't create a bound state. Here the pole  $x^+ + x_B$  is responsible for the creation of bound states. Once again, the pole is hidden in overall factors of higher order  $K$ -matrices because of normalization  $k_1 = 1$  that we use in calculations.

### The $K$ -matrix $K^{Ba}$

The  $K$ -matrix  $K^{Ba}$  has the following components  $k_i$ :

$$\begin{aligned}
k_1 &= 1, \\
k_2 &= 1 + 3 \frac{x_B \left( (y^-)^2 - (y^+)^2 \right) \left( 1 + (y^+)^2 \right)}{2 (x_B - y^-) (1 + x_B y^-) (y^+)^2}, \\
k_3 &= - \frac{\left( (y^-)^2 + x_B y^+ \right) \tilde{\eta}}{(x_B - y^-) y^- \eta}, \\
k_4 &= - \frac{(x_B + y^+) \left( y^- + x_B (y^+)^2 \right) \tilde{\eta}}{(x_B - y^-) (1 + x_B y^-) y^+ \eta}, \\
k_5 &= \frac{\left( x_B y^- - (y^+)^2 \right) \tilde{\eta}_B}{(x_B - y^-) y^- \eta_B}, \\
k_6 &= \frac{-x_B (y^-)^4 + x_B y^- y^+ + 4 (y^-)^2 y^+ + x_B (y^-)^3 y^+ + x_B (y^+)^2 - 2 x_B (y^-)^2 (y^+)^2 \tilde{\eta}^2}{2 (x_B - y^-) (y^-)^2 (1 + x_B y^-)} \frac{\tilde{\eta}^2}{\eta^2}, \\
k_7 &= - \frac{y^+ (x_B + y^+) \tilde{\eta} \tilde{\eta}_B}{(x_B - y^-) y^- \eta \eta_B}, \\
k_8 &= \frac{\left( 2 x_B^2 (y^-)^3 - x_B y^- y^+ + x_B^2 (y^-)^2 y^+ + y^- (y^+)^2 - x_B (y^-)^2 (y^+)^2 + 2 (y^+)^3 \right) \tilde{\eta} \tilde{\eta}_B}{(x_B - y^-) (y^-)^2 (1 + x_B y^-)} \frac{\tilde{\eta} \tilde{\eta}_B}{\eta \eta_B}, \\
k_9 &= \frac{y^+ (x_B + y^+) \left( -x_B (y^-)^2 + y^+ \right) \tilde{\eta}^2 \tilde{\eta}_B}{(x_B - y^-) (y^-)^2 (1 + x_B y^-)} \frac{\tilde{\eta}^2 \tilde{\eta}_B}{\eta^2 \eta_B}, \\
k_{10} &= - \frac{i \zeta x_B \left( (y^-)^2 - (y^+)^2 \right)^2}{2 (x_B - y^-) y^- (1 + x_B y^-) y^+ \eta^2}, \\
k_{11} &= - \frac{i x_B (y^- + y^+)^2 \tilde{\eta}^2}{2 \zeta (x_B - y^-) y^- (1 + x_B y^-) y^+}, \\
k_{12} &= - \frac{i \zeta x_B \left( x_B y^- - (y^+)^2 \right) \left( (y^-)^2 - (y^+)^2 \right)}{\sqrt{2} (x_B - y^-) y^- (1 + x_B y^-) y^+ \eta \eta_B}, \\
k_{13} &= \frac{i (y^- + y^+) \left( x_B y^- - (y^+)^2 \right) \tilde{\eta} \tilde{\eta}_B}{\sqrt{2} \zeta (x_B - y^-) y^- (1 + x_B y^-) y^+}, \\
k_{14} &= \frac{x_B \left( x_B (y^-)^2 - y^+ \right) (y^- + y^+) \tilde{\eta}^2}{\sqrt{2} (y^-)^2 (-x_B + y^-) (1 + x_B y^-) \eta \eta_B}, \\
k_{15} &= \frac{\left( x_B (y^-)^2 - y^+ \right) \left( (y^-)^2 - (y^+)^2 \right) \tilde{\eta} \tilde{\eta}_B}{\sqrt{2} (x_B - y^-) (y^-)^2 (1 + x_B y^-)} \frac{\tilde{\eta} \tilde{\eta}_B}{\eta^2}, \\
k_{16} &= - \frac{i (x_B + y^+) (y^- + y^+) \tilde{\eta}^2 \tilde{\eta}_B}{\sqrt{2} \zeta y^- (-x_B + y^-) (1 + x_B y^-)} \frac{\tilde{\eta}^2 \tilde{\eta}_B}{\eta},
\end{aligned}$$

$$\begin{aligned}
k_{17} &= \frac{i\zeta x_B (x_B + y^+) \left( -(y^-)^2 + (y^+)^2 \right)}{\sqrt{2} (x_B - y^-) y^- (1 + x_B y^-)} \frac{\tilde{\eta}}{\eta^2 \eta_B}, \\
k_{18} &= \frac{x_B (y^- + y^+)}{\sqrt{2} (x_B - y^-) y^-} \frac{\tilde{\eta}}{\eta_B}, \\
k_{19} &= \frac{\left( (y^-)^2 - (y^+)^2 \right)}{\sqrt{2} y^- (-x_B + y^-)} \frac{\tilde{\eta}_B}{\eta}.
\end{aligned}$$

### The K-Matrix $K^{Ab}$

The  $K$ -matrix  $K^{Ab}$  has the following components  $k_i$ :

$$\begin{aligned}
k_1 &= 1, \\
k_2 &= 1 - \frac{3 (x^- + x^+) \left( (x^-)^2 + y_B^2 (x^+)^2 \right)}{2 (y_B - x^-) x^- x^+ (-1 + y_B x^+)}, \\
k_3 &= \frac{\left( y_B x^- - (x^+)^2 \right)}{(y_B - x^-) x^-} \frac{\tilde{\eta}_B}{\eta_B}, \\
k_4 &= -\frac{(y_B + x^+) \left( x^- + y_B (x^+)^2 \right)}{(y_B - x^-) x^- (-1 + y_B x^+)} \frac{\tilde{\eta}_B}{\eta_B}, \\
k_5 &= -\frac{\left( (x^-)^2 + y_B x^+ \right)}{(y_B - x^-) x^-} \frac{\tilde{\eta}}{\eta}, \\
k_6 &= \frac{x^+ \left( y_B^2 (x^-)^3 - 2y_B x^- x^+ - y_B^2 (x^-)^2 x^+ - x^- (x^+)^2 - 2y_B (x^-)^2 (x^+)^2 + (x^+)^3 \right) \tilde{\eta}_B^2}{2 (y_B - x^-) (x^-)^3 (-1 + y_B x^+) \eta_B^2}, \\
k_7 &= -\frac{x^+ (y_B + x^+) \tilde{\eta} \tilde{\eta}_B}{(y_B - x^-) x^- \eta \eta_B}, \\
k_8 &= \frac{x^+ \left( (x^-)^2 x^+ - y_B (x^-)^2 - 2y_B (x^-)^4 - y_B^2 (x^-)^2 x^+ + 2y_B (x^+)^2 + y_B (x^-)^2 (x^+)^2 \right) \tilde{\eta} \tilde{\eta}_B}{(y_B - x^-) (x^-)^3 (-1 + y_B x^+) \eta \eta_B}, \\
k_9 &= \frac{(x^+)^2 (y_B + x^+) \left( -y_B (x^-)^2 + x^+ \right)}{(y_B - x^-) (x^-)^3 (-1 + y_B x^+)} \frac{\tilde{\eta} \tilde{\eta}_B^2}{\eta \eta_B^2}, \\
k_{10} &= -\frac{i\zeta y_B^2 (x^- - x^+) (x^- + x^+)^2}{2 (y_B - x^-) (x^-)^2 (-1 + y_B x^+) \eta_B^2}, \\
k_{11} &= -\frac{i (x^- - x^+) (x^- + x^+)^2 \tilde{\eta}_B^2}{2\zeta (y_B - x^-) (x^-)^2 (-1 + y_B x^+)}, \\
k_{12} &= \frac{i\zeta y_B \left( (x^-)^2 + y_B x^+ \right) \left( (x^-)^2 - (x^+)^2 \right)}{\sqrt{2} (y_B - x^-) (x^-)^2 (-1 + y_B x^+) \eta \eta_B},
\end{aligned}$$

$$\begin{aligned}
k_{13} &= \frac{i(x^- + x^+) \left( (x^-)^2 + y_B x^+ \right) \tilde{\eta} \tilde{\eta}_B}{\sqrt{2} \zeta (x^-)^2 (-y_B + x^-) (-1 + y_B x^+)}, \\
k_{14} &= \frac{\left( y_B (x^-)^2 - x^+ \right) x^+ \left( (x^-)^2 - (x^+)^2 \right) \tilde{\eta}_B^2}{\sqrt{2} (x^-)^3 (-y_B + x^-) (-1 + y_B x^+) \eta \eta_B}, \\
k_{15} &= \frac{y_B \left( y_B (x^-)^2 - x^+ \right) x^+ (x^- + x^+) \tilde{\eta} \tilde{\eta}_B}{\sqrt{2} (y_B - x^-) (x^-)^3 (-1 + y_B x^+) \eta_B^2}, \\
k_{16} &= \frac{i x^+ (y_B + x^+) (x^- + x^+) \tilde{\eta} \tilde{\eta}_B^2}{\sqrt{2} \zeta (x^-)^2 (-y_B + x^-) (-1 + y_B x^+) \eta_B}, \\
k_{17} &= \frac{i \zeta y_B x^+ (y_B + x^+) \left( (x^-)^2 - (x^+)^2 \right) \tilde{\eta}_B}{\sqrt{2} (y_B - x^-) (x^-)^2 (-1 + y_B x^+) \eta \eta_B^2}, \\
k_{18} &= \frac{\left( (x^-)^2 - (x^+)^2 \right) \tilde{\eta}_B}{\sqrt{2} x^- (-y_B + x^-) \eta}, \\
k_{19} &= \frac{y_B (x^- + x^+) \tilde{\eta}}{\sqrt{2} (y_B - x^-) x^- \eta_B}.
\end{aligned}$$

### The $K$ -matrix $K^{Bb}$

The  $K$ -matrix  $K^{Bb}$  has the following components  $k_i$ :

$$\begin{aligned}
k_1 &= 1, \\
k_2 &= -\frac{(y_B + y^+) \left( y^- + y_B (y^+)^2 \right)}{(y_B - y^-) (1 + y_B y^-) (y^+)^3 O_1} \\
&\quad \times \left( 3y_B (y^-)^2 + (-y_B + y^- (-2 + 2y_B^2 + y_B y^-)) (y^+)^2 - 3y_B (y^+)^4 \right), \\
k_3 &= -\frac{1}{(y_B - y^-) (1 + y_B y^-) (y^+)^3 O_1} \\
&\quad \times \left( -y_B (y^+)^4 \left( 1 + y_B^2 + 2(y^+)^2 - 2y_B (y^+)^3 \right) \right. \\
&\quad \left. + y_B (y^-)^3 \left( -2y_B + 2y_B^2 y^+ + (1 + y_B^2) (y^+)^3 \right) \right. \\
&\quad \left. + (y^-)^2 (y^+)^2 \left( -2y_B (-2 + y_B^2) + y^+ (1 - 4y_B^2 + y_B^4 + (y_B + y_B^3) y^+) \right) \right. \\
&\quad \left. - y^- (y^+)^3 (y_B + y_B^3 + y^+ (1 - 4y_B^2 + y_B^4 + (-2y_B + 4y_B^3) y^+)) \right), \\
k_4 &= -\frac{y^+ (y_B + y^+) \tilde{\eta} \tilde{\eta}_B}{(y_B - y^-) y^- \eta \eta_B},
\end{aligned}$$

$$\begin{aligned}
k_5 &= -\frac{\tilde{\eta} \tilde{\eta}_B}{(y_B - y^-) (y^-)^2 (1 + y_B y^-) y^+ O_1 \eta \eta_B} \\
&\times \left( 4y_B^3 (y^-)^4 (-1 + y_B y^+) + y^- (y^+)^3 (y_B (-5 + y_B^2) + y^+ (3 + 7y_B^2 - 2y_B y^+)) \right. \\
&\quad - 4(y^+)^5 (-1 + y_B y^+) + y_B (y^-)^3 y^+ (-2y_B^2 + y^+ (7y_B + 3y_B^3 + y^+ - 5y_B^2 y^+)) \\
&\quad \left. + (y^-)^2 y^+ (2y_B^2 + y^+ (y^+ (1 + 6y_B^2 + y_B^4 - y_B y^+ (1 + y_B^2 - 2y_B y^+)) - y_B (1 + y_B^2))) \right), \\
k_6 &= -\frac{(y_B + y^+) O_2 \tilde{\eta} \tilde{\eta}_B}{(y_B - y^-) y^- O_1 \eta \eta_B}, \\
k_7 &= \frac{(y_B + y^+) \tilde{\eta} \tilde{\eta}_B}{(y_B - y^-) (y^-)^2 (1 + y_B y^-) y^+ O_1 \eta \eta_B}, \\
&\times \left( 4y_B^2 (y^-)^4 + y^- (y^+)^3 (3 - 3y_B^2 + 2y_B y^+) + y_B (y^-)^3 y^+ (2y_B + 3(-1 + y_B^2) y^+) \right. \\
&\quad \left. + 4y_B (y^+)^5 + (y^-)^2 y^+ (1 + y_B y^+) (-2y_B + y^+ (1 + y_B^2 - 2y_B y^+)) \right), \\
k_8 &= -\frac{(y_B (y^-)^2 - y^+) (y^+)^2 (y_B + y^+) \tilde{\eta}^2 \tilde{\eta}_B^2}{(y_B - y^-) (y^-)^5 (-1 + y_B y^+) O_1 \eta^2 \eta_B^2} \\
&\times \left( y_B (y^-)^4 - y_B (y^+)^2 + (y^-)^2 (-y_B + y^+ (-2 + 2y_B^2 + y_B y^+)) \right), \\
k_9 &= \frac{y_B y^- - (y^+)^2 \tilde{\eta}_B}{(y_B - y^-) y^- \eta_B}, \\
k_{10} &= -\frac{(y^-)^2 + y_B y^+ \tilde{\eta}}{y^- (y_B - y^-) \eta}, \\
k_{11} &= \frac{2(y_B + y^+) (y_B y^- - (y^+)^2) (y^- + y_B (y^+)^2) \tilde{\eta}_B}{(y_B - y^-) y^- y^+ O_1 \eta_B}, \\
k_{12} &= \frac{2((y^-)^2 + y_B y^+) \tilde{\eta}}{(y_B - y^-) (y^-)^2 y^+ O_1 \eta} \\
&\times \left( -y_B (y^-)^2 - y^- (y_B + y^-) y^+ + (y_B + y^- + y_B (y^-)^2) (y^+)^2 - 2y_B y^- (y^+)^3 \right), \\
k_{13} &= \frac{(y_B y^- - (y^+)^2) \tilde{\eta}_B^2}{(y_B - y^-) (y^-)^3 (1 + y_B y^-) y^+ O_1 \eta_B^2}, \\
&\times \left( -y_B^3 (y^-)^3 + y_B^4 (y^-)^3 y^+ + y_B y^- (2 + y_B^2 + y_B y^- (5 + y_B^2 + y_B y^-)) (y^+)^2 \right. \\
&\quad + y^- (1 + y_B y^- (2 - 2y_B^2 + y_B y^-)) (y^+)^3 \\
&\quad \left. - (1 + y_B^2 + y_B (2 + y_B^2) y^-) (y^+)^4 + y_B^2 y^- (y^+)^5 \right), \\
k_{14} &= -\frac{((y^-)^2 + y_B y^+) (y_B (y^-)^4 - y_B (y^+)^2 + (y^-)^2 (-y_B + y^+ (-2 + 2y_B^2 + y_B y^+))) \tilde{\eta}^2}{(y_B - y^-) (y^-)^3 O_1 \eta^2}, \\
k_{15} &= -\frac{2(y_B (y^-)^2 - y^+) y^+ (y_B + y^+) ((y^-)^2 + y_B y^+) \tilde{\eta}^2 \tilde{\eta}_B}{(y_B - y^-) (y^-)^3 O_1 \eta^2 \eta_B},
\end{aligned}$$

$$\begin{aligned}
k_{16} &= -\frac{2\left(y_B(y^-)^2 - y^+\right)(y^+)^2(y_B + y^+)\tilde{\eta}\tilde{\eta}_B^2}{(y_B - y^-)(y^-)^3(-1 + y_By^+)O_1\eta\eta_B^2} \\
&\quad \times \left(y_B + y^- + y_B(y^-)^2 - (1 + y_By^-)y^+ + y_B(y^+)^2\right), \\
k_{17} &= -\frac{\zeta^2 y_B^2(y_B + y^+)\left((y^-)^2 + y_By^+\right)\left((y^-)^2 - (y^+)^2\right)^2}{(y_B - y^-)(y^-)^3(-1 + y_By^+)O_1\eta^2\eta_B^2}, \\
k_{18} &= \frac{(y_B + y^+)(y^- + y^+)^2\left(y_By^- - (y^+)^2\right)\tilde{\eta}^2\tilde{\eta}_B^2}{\zeta^2(y_B - y^-)(y^-)^2(1 + y_By^-)y^+O_1}, \\
k_{19} &= -\frac{i\zeta y_B(y_B + y^+)\left(y_By^- - (y^+)^2\right)\left((y^-)^2 - (y^+)^2\right)\left(y^- + y_B(y^+)^2\right)}{(y_B - y^-)y^-(1 + y_By^-)(y^+)^2O_1\eta\eta_B}, \\
k_{20} &= \frac{i(y_B + y^+)(y^- + y^+)\left(y_By^- - (y^+)^2\right)\left(y^- + y_B(y^+)^2\right)\tilde{\eta}\tilde{\eta}_B}{\zeta(y_B - y^-)y^-(1 + y_By^-)(y^+)^2O_1}, \\
k_{21} &= -\frac{i\left(y_B(y^-)^2 - y^+\right)(y_B + y^+)(y^- + y^+)\left(y_By^- - (y^+)^2\right)\tilde{\eta}^2\tilde{\eta}_B^2}{\zeta(y_B - y^-)(y^-)^3(1 + y_By^-)O_1\eta\eta_B}, \\
k_{22} &= -\frac{i\zeta y_B\left(y_B(y^-)^2 - y^+\right)y^+(y_B + y^+)\left((y^-)^2 + y_By^+\right)\left((y^-)^2 - (y^+)^2\right)\tilde{\eta}\tilde{\eta}_B}{(y_B - y^-)(y^-)^4(-1 + y_By^+)O_1\eta^2\eta_B^2}, \\
k_{23} &= \frac{i\zeta y_B(-1 + y_By^+)\left(-y_By^- + (y^+)^2\right)^2\left((y^-)^2 - (y^+)^2\right)}{(y_B - y^-)y^-(1 + y_By^-)(y^+)^2O_1\eta\eta_B}, \\
k_{24} &= \frac{i(y^- + y^+)(-1 + y_By^+)\left(y_By^- - (y^+)^2\right)^2\tilde{\eta}\tilde{\eta}_B}{\zeta(y_B - y^-)y^-(1 + y_By^-)(y^+)^2O_1}, \\
k_{25} &= -\frac{\left(y_B(y^-)^2 - y^+\right)\left((y^-)^2 - (y^+)^2\right)^2\tilde{\eta}_B^2}{(y_B - y^-)(y^-)^3O_1\eta^2}, \\
k_{26} &= \frac{y_B^2(y^- + y^+)^2\tilde{\eta}^2}{(y_B - y^-)(y^-)^3(1 + y_By^-)y^+O_1\eta_B^2} \\
&\quad \times \left(y_B(y^-)^4y^+ + (y^+)^2 + y_By^-(y^+)^2 - (y^-)^3(y_B + y^+(-1 + y_By^+))\right), \\
k_{27} &= -\frac{\left((y^-)^2(2 + y_B^2 + y_By^-) + (y_B + y^- + 2y_B^2y^-)y^+\right)\left((y^-)^2 - (y^+)^2\right)\tilde{\eta}_B}{(y_B - y^-)(y^-)^2O_1\eta}, \\
k_{28} &= \frac{y_B(y_B + y^+)(y^- + y^+)\left(y^- + y_B(y^+)^2\right)\tilde{\eta}}{(y_B - y^-)y^-y^+O_1\eta_B}, \\
k_{29} &= \frac{y^+(y_B + y^+)\left(-y_B(y^-)^2 + y^+\right)\left((y^-)^2 - (y^+)^2\right)\tilde{\eta}\tilde{\eta}_B^2}{(y_B - y^-)(y^-)^3O_1\eta^2\eta_B}, \\
k_{30} &= \frac{y_B\left(y_B(y^-)^2 - y^+\right)y^+(y_B + y^+)(y^- + y^+)\tilde{\eta}^2\tilde{\eta}_B}{(y_B - y^-)(y^-)^3O_1\eta\eta_B^2},
\end{aligned}$$



$$\begin{aligned}
k_{31} &= \frac{(y^+)^2 - (y^-)^2}{(y_B - y^-) y^-} \frac{\tilde{\eta}_B}{\eta}, \\
k_{32} &= \frac{y_B (y^- + y^+)}{(y_B - y^-) y^-} \frac{\tilde{\eta}}{\eta_B}, \\
k_{33} &= -\frac{(y^- + y^+) \left( y_B y^- - (y^+)^2 \right)}{y_B (y_B - y^-) (y^-)^2 (1 + y_B y^-) y^+ O_1} \frac{\tilde{\eta}_B^2}{\eta \eta_B} \\
&\quad \times \left( (y^+)^3 - y_B (y^+)^4 + y_B y^- y^+ (y_B + y^+)^2 + y_B^3 (y^-)^2 (-1 + y_B y^+) \right), \\
k_{34} &= \frac{y_B (y^- + y^+) \left( y_B y^- - (y^+)^2 \right)}{(y_B - y^-) (y^-)^3 (1 + y_B y^-) y^+ O_1} \frac{\tilde{\eta} \tilde{\eta}_B}{\eta_B^2} \\
&\quad \times \left( y_B (y^-)^4 y^+ + (y^+)^2 + y_B y^- (y^+)^2 - (y^-)^3 (y_B + y^+ (-1 + y_B y^+)) \right), \\
k_{35} &= -\frac{y_B \left( y_B (y^-)^2 - y^+ \right) (y^- + y^+) (-1 + y_B y^+) \left( y_B y^- - (y^+)^2 \right) \tilde{\eta}^2}{(y_B - y^-) (y^-)^2 (1 + y_B y^-) y^+ O_1 \eta \eta_B}, \\
k_{36} &= -\frac{\left( y_B (y^-)^2 - y^+ \right) \left( (y^-)^2 + y_B y^+ \right) \left( (y^-)^2 - (y^+)^2 \right) \tilde{\eta} \tilde{\eta}_B}{(y_B - y^-) (y^-)^3 O_1 \eta^2}, \\
k_{37} &= -\frac{i \zeta y_B^2 (y^- - y^+) (y^- + y^+)^2}{(y_B - y^-) (y^-)^2 (1 + y_B y^-) (y^+)^2 O_1 \eta_B^2} \\
&\quad \times \left( y_B (y^-)^3 y^+ + (y^+)^3 + y_B y^- (y^+)^3 - (y^-)^2 (y_B + y^+ (-1 + y_B y^+)) \right), \\
k_{38} &= -\frac{i (y^- - y^+) (y^- + y^+)^2 \left( y_B y^- - (y^+)^2 \right) (-y_B + y^+ (1 + y_B y^- - y_B y^+)) \tilde{\eta}_B^2}{\zeta y_B (y_B - y^-) y^- (1 + y_B y^-) (y^+)^2 O_1}, \\
k_{39} &= \frac{i \zeta y_B \left( (y^-)^2 + y_B y^+ \right) \left( (y^-)^2 - (y^+)^2 \right)^2}{(y_B - y^-) (y^-)^2 y^+ O_1 \eta^2}, \\
k_{40} &= -\frac{i y_B (y^- + y^+)^2 (-1 + y_B y^+) \left( y_B y^- - (y^+)^2 \right) \tilde{\eta}^2}{\zeta (y_B - y^-) y^- (1 + y_B y^-) (y^+)^2 O_1}, \\
k_{41} &= \frac{i \zeta y_B (1 + y_B y^-) y^+ (y_B + y^+) \left( (y^-)^2 + y_B y^+ \right) \left( (y^-)^2 - (y^+)^2 \right) \tilde{\eta}_B}{(y_B - y^-) (y^-)^3 (-1 + y_B y^+) O_1 \eta \eta_B^2}, \\
k_{42} &= -\frac{i (y_B + y^+) (y^- + y^+) \left( -y_B y^- + (y^+)^2 \right) \tilde{\eta} \tilde{\eta}_B^2}{\zeta (y_B - y^-) (y^-)^2 O_1 \eta_B}, \\
k_{43} &= \frac{i \zeta y_B (y_B + y^+) \left( (y^-)^2 + y_B y^+ \right) \left( (y^-)^2 - (y^+)^2 \right) \tilde{\eta}}{(y_B - y^-) (y^-)^2 O_1 \eta^2 \eta_B}, \\
k_{44} &= -\frac{i (y_B + y^+) (y^- + y^+) \left( (y^-)^2 + y_B y^+ \right) \tilde{\eta}^2 \tilde{\eta}_B}{\zeta (y_B - y^-) (y^-)^2 O_1 \eta}, \\
k_{45} &= k_{46} = k_{47} = k_{48} = 0,
\end{aligned}$$

where we have defined

$$O_1 = \frac{1}{2} \left( 2y_B + y^- - y_B^2 y^- + 2y_B (y^-)^2 - y^+ (3 + y_B y^-) + 2y_B (y^+)^2 \right),$$

$$O_2 = \left( (y^+)^2 (-1 + y_B^2 - 2y_B y^+) + y^- (2y_B + y^+ - y_B^2 y^+) \right).$$

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